A sub-Riemannian Santaló formula with applications to isoperimetric inequalities and Dirichlet spectral gap of hypoelliptic operators

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Outline

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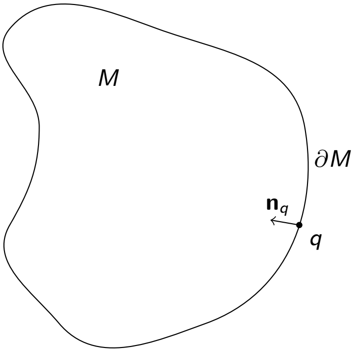
The Riemannian Santaló formula

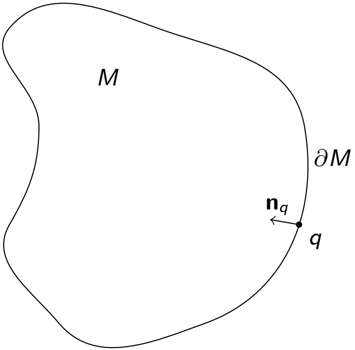
- Classical formula in integral geometry [Santaló 1976]
- It essentially allows to decompose integrals on a manifold with boundary in integrals along the geodesics starting from the boundary.
- Deep consequences as [Croke 1980–1987]:
  - Isoperimetric inequalities
  - Geometric inequalities (Hardy, Poincaré,...)
  - Lower bounds for λ_1

Riemannian Santaló formula: Setting



- $(M, \mathbf{g})$ Riemannian manifold
  - Compact
  - With boundary $\partial M \neq \varnothing$
- $\omega = \mathrm{vol}_g$ Riemannian volume form
- $\mathbf{n}$ unit inward pointing vector

Riemannian Santaló formula: Setting



- Unit tangent bundle

$$UM = \{v \in TM \mid |v| = 1\}$$

- Geodesic flow $\Phi_t : UM \to UM$,

$$\Phi_t(v) = \dot{\gamma}_v(t)$$

- Exit time $\ell(v) \in [0, +\infty]$,

$$\ell(v) := \sup\{t \geq 0 \mid \gamma_v(t) \in M\}$$

- Visible unit tangent bundle

$$U^{\text{☀}}M = \{v \in UM \mid \ell(-v) < +\infty\}$$

Riemannian Santaló formula

The *Liouville measure* $\Theta = d\dot{q} \wedge dq$ is the natural measure on TM with coordinates $(q, \dot{q})$. From it we can derive

- the *Liouville surface measure* μ on UM
- the measure η_q on fibers above q, that is,

$$\int_{UM} F \, d\mu = \int_M \left(\int_{U_q M} F(q, \dot{q}) \, d\eta_q(\dot{q}) \right) \, d\omega(q)$$

In coordinates, η is the standard measure on $U_q M \cong \mathbb{S}^{n-1}$.

Theorem (Santaló formula)

$F : UM \to \mathbb{R}$ measurable function

$$\int_{U^{\Sunrise} M} F \, d\mu = \int_{\partial M} \int_{U_q^+ \partial M} \left(\int_0^{\ell(v)} F(\Phi_t(v)) \, dt \right) \mathbf{g}(v, \mathbf{n}_q) \, d\eta(v) \, d\sigma(q).$$

Here, $U_q^+ M = \{ v \in U_q M \mid \mathbf{g}(v, \mathbf{n}_q) \geq 0 \}$ is the set of inward pointing unit vectors on the boundary.

Consequences

Choosing $F \equiv 1$ in Santaló formula one gets,

Theorem (Croke)

Letting $\vartheta^{☀} \in [0,1]$ *is the visibility angle of* M*, it holds*

$$\frac{\sigma(\partial M)}{\omega(M)} \geq \frac{2\pi|\mathbb{S}^{n-1}|}{|\mathbb{S}^n|} \frac{\vartheta^{☀}}{diam(M)}.$$

Equality holds if and only if M *is isometric to an hemisphere.*

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For any $f : M \to \mathbb{R}$ one can lift it to $F : UM \to \mathbb{R}$ as $F(q, \dot{q}) = f(q)$.

Theorem (Croke, Derdzinski)

For any $f \in C_0^\infty(M)$ *it holds*

$$\int_M |\nabla f|^2 \, d\omega \geq \frac{n\pi^2}{|\mathbb{S}^{n-1}|L^2} \int_M |f|^2 \, d\omega \implies \lambda_1(M) \geq \frac{n\pi^2}{|\mathbb{S}^{n-1}|L^2},$$

where $L \leq +\infty$ *is the length of the longest Riemannian geodesic contained in* M*. The equality for* $\lambda_1(M)$ *holds if and only if* M *is isometric to an hemisphere.*

Outline

Sub-Riemannian geometry

Definition

A *sub-Riemannian manifold* is a triple $(M, \Delta, \mathbf{g})$, where

(*i*) M is a connected smooth manifold of dimension $n \geq 3$;

(*ii*) Δ is a smooth distribution of constant rank $k < n$, i.e. a smooth map that associates to $q \in M$ a k-dimensional subspace Δ_q of T_qM satisfying the *Hörmander condition*.

$$\operatorname{span}\{[X_1, [\dots[X_{j-1}, X_j]]](q) \mid X_i \in \overline{\Delta}, j \in \mathbb{N}\} = T_qM, \quad \forall q \in M,$$

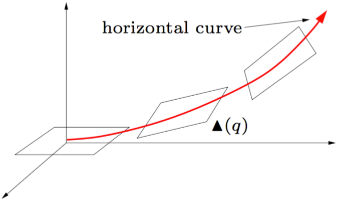
where $\overline{\Delta}$ denotes the set of *horizontal smooth vector fields* on M, i.e.

$$\overline{\Delta} = \{X \in \mathrm{Vec}(M) \mid X(q) \in \Delta_q \ \ \forall q \in M\}.$$

(*iii*) $\mathbf{g}_q$ is a Riemannian metric on Δ_q, that is smooth as function of q.

In the following,

- M is compact with boundary $\partial M \neq \varnothing$.
- A smooth volume ω is fixed on M.



Sub-Riemannian distance:

$$d(p,q) = \inf \left\{ \int_0^T g(\dot\gamma(t),\dot\gamma(t))\,dt \mid \dot\gamma(t) \in \Delta_{\gamma(t)} \text{ and } \begin{array}{c} \gamma(0)=p \\ \gamma(T)=q \end{array} \right\}.$$

Remark

Thanks to the Hörmander condition we have that (M,d) is a metric space with the same topology of the original one of M (Chow, Rashevsky theorem)

Connection with control theory

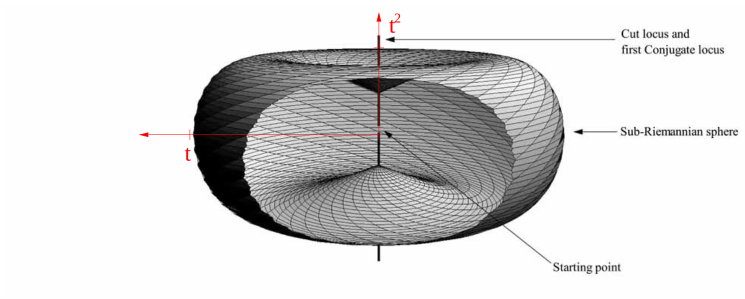
Locally, the pair $(\Delta, \mathbf{g})$ can be given by assigning a set of k smooth vector fields (called a *local orthonormal frame*) spanning Δ and that are orthonormal for $\mathbf{g}$, i.e.

$$\Delta_q = \mathrm{span}\{X_1(q), \dots, X_k(q)\}, \qquad \mathbf{g}_q(X_i(q), X_j(q)) = \delta_{ij}.$$

The problem of finding the curve of minimal length between two given points q_0, q_1, becomes the optimal control problem

$$\begin{cases} \dot{q}(t) = \sum_{i=1}^{k} u_i(t) X_i(q(t)) \\ \int_0^T \sqrt{\sum_{i=1}^{k} u_i^2(t)} \longrightarrow \min \\ q(0) = q_0, \quad q(T) = q_1 \end{cases}$$

Basic features in SRG



- Spheres are highly non-isotropic
- there are geodesics loosing optimality close to the starting point
  $\Longrightarrow$ spheres are never smooth even for small time
- the Hausdorff dimension is always bigger than the topological dimension

Difficulties to the extension

- **Boundary:** The sR normal is the inward-pointing unit vector $\mathbf{n}_q \in \Delta_q$ such that
$$\mathbf{n}_q \perp v \quad \forall v \in T_q\partial M \cap \Delta_q.$$
If Δ_q is tangent to ∂M, $\mathbf{n}_q$ is not well-defined!

  (H0) The set of points $q \in \partial M$ such that Δ_q is tangent to ∂M is negligible.

- **Geodesic flow:** Since initial velocities of geodesics are constrained to $\Delta \subset TM$, more than one geodesic can start with the same one.

  (!!!) There is no geodesic flow $\Phi_t : TM \to TM$.

Geodesics in sub-Riemannian geometry

- **Hamiltonian formulation:** Consider the Hamiltonian $H : T^*M \to \mathbb{R}$

$$H(q,p) = \frac{1}{2}\sum_{i=1}^{k} \langle p, X_i(q)\rangle^2.$$

Theorem (Pontryagin Maximum Principle for normal extremals)

For any $\lambda_0 \in T^*M$, the solution $\lambda : [0,T] \to T^*M$ with $\lambda(0) = \lambda_0$ of the Hamiltonian system

$$\dot{\lambda}(t) = \vec{H}(\lambda(t)), \qquad \vec{H} = \frac{\partial H}{\partial q}\frac{\partial}{\partial p} + \frac{\partial H}{\partial p}\frac{\partial}{\partial q},$$

projects to a minimizer $\gamma = \pi \circ \lambda : [0,T] \to M$ of the sR distance.

- Normal minimizers parametrized by the initial covector
- Sub-Riemannian geodesic flow $\Phi_t : T^*M \to T^*M$

Hamiltonian Santaló formula

We consider the sub-Riemannian geodesic flow $\Phi_t : U^*M \to U^*M$ with

$$U^*M = \left\{ \lambda \in T^*M \mid H(\lambda) = \frac{1}{2} \right\} \cong \mathbb{R}^{n-k} \times \mathbb{S}^{k-1}.$$

The Liouville measure on the cotangent bundle T^*M still allows to define

- a Liouville surface measure μ on U^*M,
- a "vertical measure" η_q on the fibers U_q^*M.

Theorem

Let $F : U^*M \to \mathbb{R}$ measurable. Then

$$\int_{U^*M} F \, d\mu = \int_{\partial M} \int_{U_q^+\partial M} \left(\int_0^{\ell(\lambda)} F(\Phi_t(v)) \, dt \right) \langle \lambda, \mathbf{n}_q \rangle \, d\eta_q(\lambda) \, d\sigma(q).$$

Here, $U_q^+M = \{\lambda \in U_q^*M \mid \langle \lambda, \mathbf{n}_q \rangle \geq 0\} \subset U_q^*M$ is the set of inward pointing unit covectors on the boundary.

- **Problem:** The typical fiber U_q^*M is not compact!
  $\Longrightarrow$ Choosing $F \equiv 1$ or lifting $f : M \to \mathbb{R}$ as before gives an infinite result on both sides!

Outline

1 Riemannian Santaló formula

2 Sub-Riemannian Santaló formula

3 Reduction procedure

Reduction technique

First proposed by [Pansu 1985] in the Heisenberg group and [Chanillo, Young 2009] for 3D Sasakian manifolds.

- Assume a complement $\mathcal{V} \subset TM$ such that $TM = \Delta \oplus \mathcal{V}$ to be fixed.
  $\implies$ Equivalent to fix a Riemannian metric $\hat{g}$ on M such that $\hat{g}|_\Delta = g$ and $\mathrm{vol}_g = \omega$.
  $\implies$ $T^*M = \Delta^\perp \oplus \mathcal{V}^\perp$.
- The *reduced cotangent bundle* is $T^*M^{\mathrm{r}} = T^*M \cap \mathcal{V}^\perp$ and $U^*M^{\mathrm{r}} = U^*M \cap \mathcal{V}^\perp$.

Example: In the Heisenberg group, given by the two vector fields on $\mathbb{R}^3$

$$X = \begin{pmatrix} 1 \\ 0 \\ -\frac{1}{2}y \end{pmatrix} \qquad Y = \begin{pmatrix} 0 \\ 1 \\ \frac{1}{2}x \end{pmatrix},$$

the geodesics with covectors in $U^*_qM^{\mathrm{r}}$ at (x, y, z) span the plane (Euclidean-) orthogonal to $(y/2, -x/2, 1)$.

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Theorem

Under some stability assumptions on the complement $\mathcal{V}$, for any $F : U^*M^{\mathrm{r}} \to \mathbb{R}$ measurable it holds

$$\int_{U^* M^{\mathrm{r}}} F\,d\mu = \int_{\partial M} \int_{U_q^+ \partial M^{\mathrm{r}}} \left(\int_0^{\ell(\lambda)} F(\Phi_t(v))\,dt \right) \langle \lambda, \mathbf{n}_q \rangle \,d\eta_q(\lambda)\,d\sigma(q). \tag{1}$$

- Now $U_q^* M^{\mathrm{r}} \cong \mathbb{S}^{k-1}$ is **compact**!

Consequences: Isoperimetric inequality

Theorem

Letting $\vartheta^{\star} \in [0,1]$ be the (reduced) visibility angle of M, it holds

$$\frac{\sigma(\partial M)}{\omega(M)} \geq \frac{2\pi|\mathbb{S}^{k-1}|}{|\mathbb{S}^k|} \frac{\vartheta^{\star}}{diam^{\mathsf{r}}(M)},$$

where $diam^{\mathsf{r}}(M)$ is the reduced sR diameter of M.

- It always holds $\mathrm{diam}^{\mathsf{r}}(M) \leq \mathrm{diam}(M)$.
- The reduced diameter is much easier to compute than $\mathrm{diam}(M)$.
- To get rid of the diameter on needs curvature arguments in Riemannian $\implies$ not available right now in the sR setting

Consequences: Poncaré-type inequality

The sub-Riemannian gradient of $f \in C^\infty(M)$ is, for a local orthonormal frame,

$$\nabla_{\mathrm{sR}} f = \sum_{i=1}^{k} (X_i f)\, X_i$$

Theorem (Poincare inequality)

For $f \in C_0^\infty(M)$ we have

$$\int_M |\nabla_{sR} f|^2 \, d\omega \geq \frac{k\pi^2}{L^2} \int_M f^2 \, d\omega,$$

where L is the length of the longest **reduced** *geodesic.*

- Similarly, we obtain (p-)Hardy-like inequality for $p > 1$.

Consequences: Spectral gap for hypoelliptic operators

The Dirichlet sub-Laplacian is the (Friedrichs extension) of the operator $\mathcal{L}$ s.t.

$$\int_M \mathbf{g}(\nabla_{\mathsf{sR}} f, \nabla_{\mathsf{sR}} g)\, d\omega = \int_M (-\mathcal{L} f) g\, d\omega \qquad \forall f, g \in C_c^\infty(M).$$

- By Hörmander condition, $-\mathcal{L}$ is hypoelliptic.
- For a local orthonormal frame, we have

$$\mathcal{L} = \sum_{i=1}^k X_i^2 + \text{first order terms}.$$

- M compact $\implies$ $\mathrm{spec}(-\mathcal{L}) = \{0 < \lambda_1(M) \le \lambda_2(M) \le \ldots \to +\infty\}$

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Theorem (Universal hypoelliptic spectral gap)

$$\lambda_1(M) \geq \frac{k\pi^2}{L^2}.$$

Final remarks

1. These results apply notably to two large classes of sub-Riemannian manifolds
   - Riemannian foliations with totally geodesic fibers
     (submersions, contact manifold with symmetries, CR manifolds, quasi-contact manifolds)
   - All Carnot groups
     (left-invariant nilpotent structures on $\mathbb{R}^n$)
2. The isoperimetric inequality and the lower spectral bound are sharp for the hemispheres of the complex and quaternionic Hopf fibrations on the spheres (generalizes the sharpness on hemispheres of Croke)
3. A (new) refinement of this technique yields better bounds on $\lambda_1(M)$, which are sharp on Riemannian cubes. Numerical results suggests the sharpness also for cubes in the Carnot cases.