Hardy-type inequalities and spectral bounds for hypoelliptic operators of Hörmander type.

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Abstract

We present (p-)Hardy-type inequalities for hypoelliptic operators Δ satisfying the Hörmander condition on compact domains M with Lipschitz boundary ∂M and negligible characteristic set. As a consequence, we prove a universal lower bound for the first Dirichlet eigenvalue λ_1 of Δ in terms only of the rank k of the associated sub-Riemannian distribution and the length L of the longest sub-Riemannian geodesic contained in M. All our results are sharp for the standard hypoelliptic operator of CR and quaternionic contact geometries on the hemispheres of the complex and quaternionic Hopf fibrations $\mathbb{S}^1 \hookrightarrow \mathbb{S}^{2d+1} \xrightarrow{p} \mathbb{CP}^d$, $\mathbb{S}^3 \hookrightarrow \mathbb{S}^{4d+3} \xrightarrow{p} \mathbb{HP}^d$ for all $d \geq 1$, where $L = \pi$.

These results are obtained via a novel sub-Riemannian version of the classical Santaló formula: a result in integral geometry that describes the intrinsic Liouville measure on the unit cotangent bundle in terms of the geodesic flow. Our construction works under quite general conditions on the sub-Riemannian structure associated with Δ, satisfied by any Carnot group and any sub-Riemannian structure associated with a Riemannian foliation with totally geodesic leaves (e.g. CR and quaternionic contact manifolds with symmetries).

Introduction

Let (M, g) be a compact Riemannian manifold with boundary ∂M. Santaló formula [5] is a classical result in integral geometry that describes the Liouville measure μ of the unit tangent bundle UM in terms of the geodesic flow $\phi_t : UM \to UM$. Namely, for any measurable function $F : UM \to \mathbb{R}$ we have

$$\int_{U^{\curlyvee}M} F\, d\mu = \int_{\partial M} \left[\int_{U_q^+\partial M} \left(\int_0^{\ell(v)} F(\phi_t(v))dt \right) g(v, \mathbf{n}_q) \eta_q(v) \right] \sigma(q), \tag{1}$$

where σ is the surface form on ∂M induced by the inward pointing normal vector $\mathbf{n}$, η_q is the Riemannian spherical measure on U_qM, $U_q^+\partial M$ is the set of inward pointing unit vectors at $q \in \partial M$ and $\ell(v)$ is the exit length of the geodesic with initial vector v. Finally, $U^{\curlyvee}M \subseteq UM$ is the visible set, i.e. the set of unit vectors that can be reached via the geodesic flow starting from points on ∂M.

In the Riemannian setting, (1) allows to deduce some very general isoperimetric inequalities, Hardy-like inequalities, and Dirichlet eigenvalues estimates for the Laplace-Beltrami operator [1, 2].

Assume now $n \geq 3$ and that a volume ω has been fixed on M. We consider the hypoelliptic operator Δ whose local expression is

$$\Delta = \sum_{i=1}^{k} X_i^2 + (\operatorname{div}_\omega X_i)X_i, \tag{2}$$

for some family of smooth linearly-independent vector fields $\mathcal{X} = \{X_1, \dots, X_k\}$ satisfying the Hörmander condition (i.e., whose repeated Lie brackets evaluated at every $x \in M$ span T_xM).

The operator Δ naturally defines a sub-Riemannian structure $(\mathcal{D}, g)$ on M, where $\mathcal{D} \subset TM$ is the bracket-generating distribution of constant rank $k \leq n$ spanned by $\mathcal{X}$ and g is the smooth metric on $\mathcal{D}$, such that $g(X_i, X_j) = \delta_{ij}$. In the following we present the extension of (1) to the most general class of sub-Riemannian structures (and hence of hypoelliptic operators) for which Santaló formula makes sense [4]. As an application we deduce Hardy-like inequalities, sharp universal estimates on the first Dirichlet eigenvalue for Δ and sharp isoperimetric-type inequalities.

This extension and its consequences are not straightforward for a number of reasons. In particular, some notable differences w.r.t. the Riemannian case are the following:

1. the geodesic flow on TM is replaced by a degenerate Hamiltonian flow on the cotangent bundle T^*M;
2. the unit cotangent bundle U^*M (the set of covectors with unit norm) is not compact, but rather has the topology of an infinite cylinder;

1 Setting

Given the sub-Riemannian structure $(\mathcal{D}, g)$ associated with Δ the sub-Riemannian distance on M is

$$d(x, y) = \inf \left\{ \int_0^1 g(\dot\gamma(t), \dot\gamma(t))\, dt \mid \gamma(0) = x, \gamma(1) = y, \dot\gamma \subset \mathcal{D} \right\}. \tag{3}$$

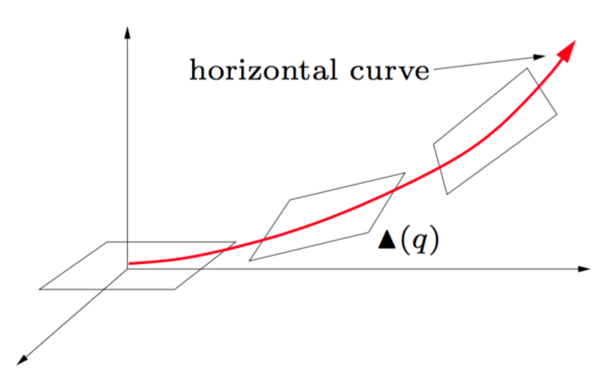

Figure 1: An *horizontal curve* with $\dot\gamma \subset \mathcal{D}$.

A *geodesic* is a curve that locally minimizes the sub-Riemannian distance between endpoints. Thanks to the Pontryiagin Maximum Principle, the geodesic flow is a natural *Hamiltonian flow* $\phi_t : T^*M \to T^*M$ on the cotangent bundle, induced by the Hamiltonian function $H \in C^\infty(T^*M)$. The latter is a non-negative, degenerate, quadratic form on the fibers of T^*M that contains all the information on the sub-Riemannian structure. Length-parametrized geodesics are characterized by an initial covector λ in the unit cotangent bundle $U^*M = \{\lambda \in T^*M \mid 2H(\lambda) = 1\}$ whose fibers are the cylinders $U_x^*M = \mathbb{S}^k \times \mathbb{R}^{n-k}$.

For any $q \in \partial M$ at which $\mathcal{D}_q$ is not tangent to ∂M, we have a well defined *inner pointing unit horizontal vector* $\mathbf{n}_q \in \mathcal{D}_q$. The *surface measure* $\sigma = \iota_{\mathbf{n}}\omega$ on ∂M is given by the contraction with the horizontal unit normal $\mathbf{n}$ to ∂M. For what concerns the regularity of the boundary, we assume only that

- ∂M is Lipschitz;
- the set of characteristic points, where $\mathcal{D}_p \subseteq T_p\partial M$, has zero measure on ∂M

1.1 Reduction procedure

A key ingredient for our results is the following *reduction procedure*, inspired by [3]. Fix a transverse sub-bundle $\mathcal{V} \subset TM$ such that $TM = \mathcal{D} \oplus \mathcal{V}$. We define the *reduced cotangent bundle* $T^*M^{\mathfrak{r}}$ as the set of covectors annihilating $\mathcal{V}$. This allows to reduce the non-compact U^*M to a compact slice $U^*M^{\mathfrak{r}} := U^*M \cap T^*M^{\mathfrak{r}}$, where we can define a reduced Liouville measure $\mu^{\mathfrak{r}}$. These must satisfy the following stability hypotheses:

(H1) The bundle $T^*M^{\mathfrak{r}}$ is invariant under the Hamiltonian flow ϕ_t;

(H2) The reduced Liouville measure is invariant, i.e. $\mathcal{L}_{\vec{H}}\mu^{\mathfrak{r}} = 0$.

These hypotheses are verified for:

- any Riemannian structure, equipped with the Riemannian volume;
- any left-invariant sub-Riemannian structure on a Carnot group, equipped with the Haar volume.
- any sub-Riemannian structure associated with a Riemannian foliation with totally geodesic leaves, equipped with the Riemannian volume (including contact, CR, QC structures with transverse symmetries).

Example 1. Important structures satisfying our assumptions are the complex and the quaternionic Hopf fibrations. Respectively,

$$\mathbb{S}^1 \hookrightarrow \mathbb{S}^{2d+1} \xrightarrow{p} \mathbb{CP}^d, \qquad \mathbb{S}^3 \hookrightarrow \mathbb{S}^{4d+3} \xrightarrow{p} \mathbb{HP}^d, \qquad d \geq 1. \tag{4}$$

Here $\mathbb{CP}^d$ and $\mathbb{HP}^d$ are the complex and quaternion projective spaces of real dimensions $2d$ and $4d$, respectively. The sub-Riemannian structure is given by considering $\mathcal{D} := (\ker p_*)^\perp$ and the sub-Riemannian metric g as the restriction to $\mathcal{D}$ of the round one.

2 Sub-Riemannian reduced Santaló formula

Notation 1. Consider a sub-Riemannian geodesic $\gamma(t)$ with initial covector $\lambda \in U^*M$.

- The *exit length* $\ell(\lambda) \in [0, +\infty)$ is the length after which γ leaves M by crossing ∂M.
- The *visible unit cotangent bundle* $U^{\curlyvee}M \subset U^*M$ is the set of unit covectors λ such that $\ell(-\lambda) < +\infty$. (See Fig. 2.)
- For any non-characteristic point $q \in \partial M$, we let $U_q^+\partial M \subset U_q^*M = \{\lambda \in U_q^*M \mid \langle\lambda, \mathbf{n}_q\rangle > 0\}$.

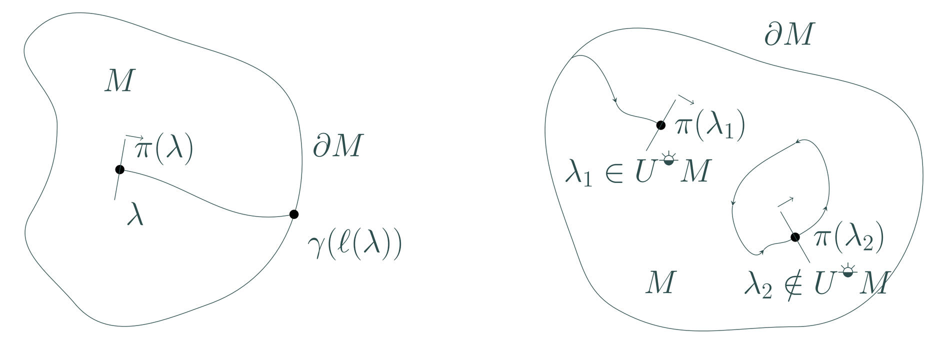

Figure 2: Exit length (left) and visible set (right). Covectors are represented as hyperplanes, the arrow shows the direction of propagation of the associated geodesic for positive time.

Theorem 1 (Reduced Santaló formula). *For any measurable function $F : U^*M^{\mathfrak{r}} \to \mathbb{R}$ we have*

$$\int_{U^{\curlyvee}M^{\mathfrak{r}}} F\, \mu^{\mathfrak{r}} = \int_{\partial M} \left[\int_{U_q^+\partial M^{\mathfrak{r}}} \left(\int_0^{\ell(\lambda)} F(\phi_t(\lambda))dt \right) \langle\lambda, \mathbf{n}_q\rangle \eta_q^{\mathfrak{r}}(\lambda) \right] \sigma(q). \tag{5}$$

Here, $\mu^{\mathfrak{r}}$ is a reduced invariant Liouville measure on $U^*M^{\mathfrak{r}}$, $\eta_q^{\mathfrak{r}}$ is an appropriate measure on $U_q^*M^{\mathfrak{r}}$.

3 Consequences

3.1 Hardy-like inequalities

For any $f \in C^\infty(M)$, let $\nabla_H f \in \Gamma(\mathcal{D})$ be the *horizontal gradient*: the horizontal direction of steepest increase of f. For a local orthonormal frame $\{X_1, \dots, X_k\}$ it is defined by

$$\nabla_H f = \sum_{i=1}^{k} (X_i f) X_i. \tag{6}$$

Proposition 2 (Hardy-like inequalities). *For any $f \in C_0^\infty(M)$ it holds*

$$\int_M |\nabla_H f|^2 \omega \geq \frac{k\pi^2}{|\mathbb{S}^{k-1}|} \int_M \frac{f^2}{R^2}\omega, \tag{7}$$

where $k = \operatorname{rank}\mathcal{D}$ and, letting $L(\lambda) := \ell(\lambda) + \ell(-\lambda)$ be the length of the maximal (reduced) geodesic that passes through q with covector λ, the function $R : M \to \mathbb{R}$ is:

$$\frac{1}{R^2(q)} := \int_{U_q^*M^{\mathfrak{r}}} \frac{1}{L^2}\eta_q^{\mathfrak{r}}, \qquad \forall q \in M. \tag{8}$$

3.2 Spectral gap for the Dirichlet spectrum

It is well-known that the Dirichlet spectrum of $-\Delta$ on the compact manifold M is positive and discrete. We have the following universal lower bound for the first Dirichlet eigenvalue $\lambda_1(M)$ on the given domain. Here, with universal we mean an estimate not requiring any *a priori* assumption on curvature or capacity.

Proposition 3 (Universal spectral lower bound). *Let $L = \sup_{\lambda \in U^*M^{\mathfrak{r}}} L(\lambda)$ be the length of the longest reduced geodesic contained in M. Then, letting $k = \operatorname{rank}\mathcal{D}$,*

$$\lambda_1(M) \geq \frac{k\pi^2}{L^2}. \tag{9}$$

Moreover, in the following cases we have equality, for all $d \geq 1$:

(i) the hemispheres $\mathbb{S}_+^d$ of the Riemannian round sphere $\mathbb{S}^d$;

(ii) the hemispheres $\mathbb{S}_+^{2d+1}$ of the sub-Riemannian complex Hopf fibration $\mathbb{S}^{2d+1}$;

(iii) the hemispheres $\mathbb{S}_+^{4d+3}$ of the sub-Riemannian quaternionic Hopf fibration $\mathbb{S}^{4d+3}$,

all equipped with the Riemannian volume of the corresponding round sphere. In all these cases, the associated eigenfunction is $\Psi = \cos(\delta)$, where δ is the Riemannian distance from the north pole.

References

[1] Christopher B. Croke. Some isoperimetric inequalities and eigenvalue estimates. *Ann. Sci. École Norm. Sup. (4)*, 13(4):419–435, 1980.

[2] Christopher B. Croke and Andrzej Derdziński. A lower bound for λ_1 on manifolds with boundary. *Comment. Math. Helv.*, 62(1):106–121, 1987.

[3] P. Pansu. An isoperimetric inequality on the Heisenberg group. *Rend. Sem. Mat. Univ. Politec. Torino*, (Special Issue):159–174 (1984), 1983. Conference on differential geometry on homogeneous spaces (Turin, 1983).

[4] D. Prandi, L. Rizzi, and M. Seri. A sub-Riemannian Santaló formula with applications to isoperimetric inequalities and Dirichlet spectral gap of hypoelliptic operators. *ArXiv e-prints*, 2015.

[5] Luis A. Santaló. *Integral geometry and geometric probability*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 2004. With a foreword by Mark Kac.