# Geometry and analysis of control-affine systems: motion planning, heat and Schrödinger evolution

Dario Prandi

Supervisor: Ugo Boscain

Ph.D course in Applied Mathematics

# Framework

We will consider affine-control systems, i.e., systems in the form

$$\dot{q}(t) = f_0(q(t)) + \sum_{i=1}^{m} u_i(t) f_i(q(t))$$

Here,

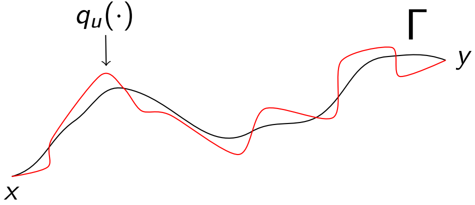
- the point $q$ belongs to a smooth manifold $M$
- the $f_i$'s are smooth vector fields on $M$
- $u \in \mathrm{L}^1([0, T], \mathbb{R}^m)$

This type of system appears in many applications

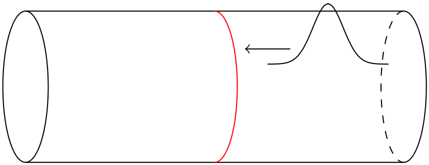
- Mechanical systems
- Quantum control
- Microswimmers
- Neuro-geometry of vision (Mumford, Petitot)

# Plan of the thesis

1. *Tracking of non-admissible curves*




2. *Diffusions on manifolds with singularities*



# Outline

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# Motion Planning

Let us consider a control system in the form

$$\dot{q}(t) = f(q(t), u(t)), \qquad u \in \mathcal{U},\ f : M \times \mathcal{U} \to TM.$$

**Problem**

Given $x, y \in M$, find an admissible trajectory steering the system from x to y, possibly under some constraints.

Possible constraints:

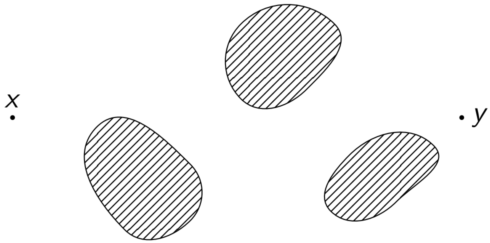
1. Avoiding some obstacles
2. Rendez-vous problem, i.e., being near certain places at certain times

**Assumption**

A metric with balls $B(q, \varepsilon)$ is fixed on M.

# Method

Different approaches are possible. We consider the following method:



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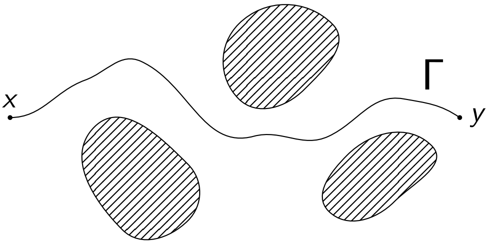
Different approaches are possible. We consider the following method:

1. Find an (non-admissible) curve $\Gamma \subset M$ or a path $\gamma : [0, T] \to M$ solving the problem.



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Different approaches are possible. We consider the following method:

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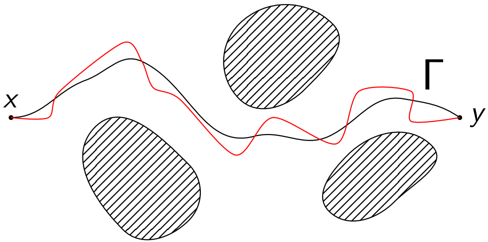
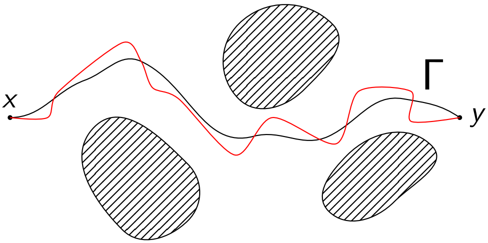
2. Track $\Gamma$ or $\gamma$ with an admissible trajectory.



# Method

Different approaches are possible. We consider the following method:

1. Find an (non-admissible) curve $\Gamma \subset M$ or a path $\gamma : [0, T] \to M$ solving the problem. $\to$ **global topology**
2. Track $\Gamma$ or $\gamma$ with an admissible trajectory. $\to$ **local behavior of the control system**




**We focus on quantifying the difficulty of the second step.**

# Complexity

Let $J : \mathcal{U} \to [0, +\infty)$ be a cost function.

**Definition (Complexity)**

A measure of the cost of approximation of a given curve/path with a certain precision

In general:

1. we fix a set $\mathrm{Adm}(\Gamma, \varepsilon)$ of admissible controls for precision $\varepsilon$
2. we define complexity as

$$\sigma(\gamma, \varepsilon) = \inf_{u \in \mathrm{Adm}(\Gamma,\varepsilon)} \frac{\text{cost of } u}{\text{cost of an ``}\varepsilon\text{ piece'' of } u} = \frac{1}{\varepsilon} \inf_{u \in \mathrm{Adm}(\Gamma,\varepsilon)} J(u, T).$$

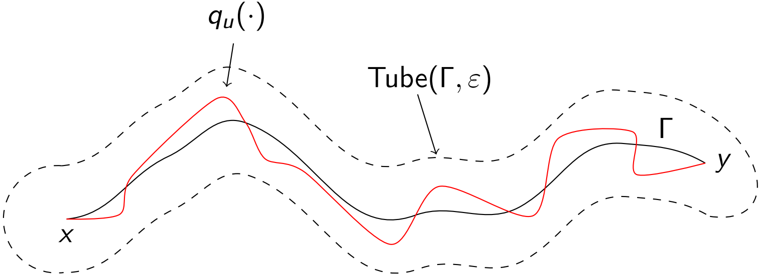
# Obstacle-avoidance problem

Let $\Gamma \subset M$ be a curve, $\text{Tube}(\Gamma, \varepsilon) = \bigcup_{q \in \gamma} B(q, \varepsilon)$, and

$$\mathcal{A}(\Gamma, \varepsilon) = \left\{ u \in \mathrm{L}^1([0, T], \mathbb{R}^m)) \mid \begin{array}{l} T > 0,\ q_u(T) = y, \\ q_u(\cdot) \subset \text{Tube}(\Gamma, \varepsilon) \end{array} \right\}.$$

With this set we define the *tubular approximation complexity*

$$\Sigma_a(\Gamma, \varepsilon) = \frac{1}{\varepsilon} \inf_{u \in \mathcal{A}(\Gamma, \varepsilon)} \mathrm{J}(u, T).$$



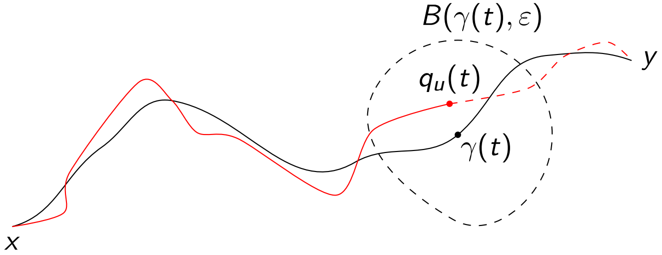
# Rendez-vous problem

Let $\gamma : [0, T] \to M$ be a path and

$$\mathcal{N}(\gamma, \varepsilon) = \left\{ u \in \mathrm{L}^1([0, T], \mathbb{R}^m)) \mid \begin{array}{l} q_u(T) = y \text{ and } q_u(t) \in B(\gamma(t), \varepsilon) \\ \text{for any } t \in [0, T] \end{array} \right\}.$$

This set defines the *neighborhood approximation complexity*

$$\sigma_n(\gamma, \varepsilon) = \frac{1}{\varepsilon} \inf_{u \in \mathcal{N}(\gamma, \varepsilon)} J(u, T).$$



# Particular case: sub-Riemannian control systems

A sub-Riemannian (or nonholonomic) control system is a control-affine system without drift

$$\dot{q}(t) = \sum_{i=1}^{m} u_i(t) f_i(q(t)),$$

that satisfies the Hörmander condition, i.e., such that

$$\mathrm{Lie}_q\{f_1, \dots, f_m\} = T_q M, \qquad \text{for any } q \in M.$$

1. The value function associated to this system w.r.t. the $\mathrm{L}^1$ cost is a distance, called sub-Riemannian distance.
2. Due to the linearity of the system, we can always reparametrize trajectories without changing their cost. Hence,

Tubular approximation complexity $\Longleftrightarrow$ Neighborhood approximation complexity

# Sub-Riemannian complexities

- Introduced by Gromov (1996) in a different context.
- *Weak equivalence:*

$$\sigma(\Gamma,\varepsilon) \asymp g(\varepsilon) \iff C_1 \le \frac{\sigma(\Gamma,\varepsilon)}{g(\varepsilon)} \le C_2 \quad \text{for } \varepsilon \downarrow 0.$$

Studied by Jean (2003).

- *Strong equivalence:*

$$\sigma(\Gamma,\varepsilon) \simeq g(\varepsilon) \iff \lim_{\varepsilon\downarrow 0} \frac{\sigma(\Gamma,\varepsilon)}{g(\varepsilon)} = 1.$$

Studied in a series of paper by Gauthier, Zakalyukin, et al.

# Control-affine systems

Recall the general form of a control-affine system

$$\dot{q}(t) = f_0(q(t)) + \sum_{i=1}^{m} u_i(t) f_i(q(t)).$$

We will consider:

- *strong Hörmander condition:* $\mathrm{Lie}_q\{f_1, \dots, f_m\} = T_q M$ for any $q \in M$.
- The set of controls is
$$\mathcal{U} = \bigcup_{t \in (0,T]} \mathrm{L}^1([0,T], \mathbb{R}^m).$$
- The cost J is the $\mathrm{L}^1$-norm of $u$.

Consequences:

1. The associated driftless system ($f_0 = 0$) is a sub-Riemannian system.
2. Small time local controllability.

# Complexities for control-affine systems

- We will use the sub-Riemannian metric to define the complexities.
- Since the system is not linear, we cannot reparametrize the trajectories, and hence

Tubular approximation complexity $\not\Longleftrightarrow$ Neighborhood approximation complexity

For any $q \in M$, $s \in \mathbb{N}$, let

$$\Delta^s(q) = \operatorname{span}\{[f_{i_1}, [f_{i_2}, [\dots, f_{i_k}] \dots]](q) \mid 1 \leq k \leq s,\ 1 \leq i_j \leq m\}.$$

$$\Delta^1(q) \subset \Delta^2(q) \subset \dots \subset \Delta^r(q) = T_q M$$

**Hypothesis**

*Equiregularity:* for any $s \in \mathbb{N}$, $\dim \Delta^s$ does not depend on the point $q \in M$.

**Theorem (F. Jean, D. P.)**

*Let* $f_0 \subset \Delta^s \setminus \Delta^{s-1}$.

- *Let* $\Gamma \subset M$ *be a smooth curve. Let* $k$ *such that* $T\Gamma \subset \Delta^k$ *and* $T\Gamma \not\subset \Delta^{k-1}$. *Then, if* $\mathcal{T}$ *is sufficiently small, we have*

$$\Sigma_a(\Gamma, \varepsilon) \asymp \frac{1}{\varepsilon^k}$$

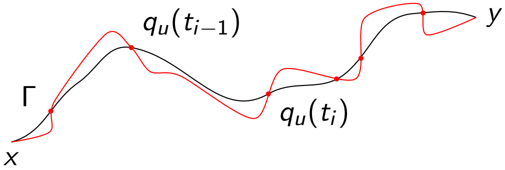
- *Let* $\gamma : [0, T] \to M$ *be a path and* $k$ *such that* $\dot\gamma \in \Delta^k$ *and* $\dot\gamma \notin \Delta^{k-1}$. *If, moreover,* $s = k$, *we assume that* $\dot\gamma \neq f_0(\gamma) \mod \Delta^{s-1}(\gamma)$. *Then*

$$\sigma_n(\gamma, \varepsilon) \asymp \frac{1}{\varepsilon^{\max\{s,k\}}}$$

- The complexity of **curves** is not sensible to the drift.
- The complexity of **paths** depends on the drift. In particular, when $f_0 \subset \Delta^r \setminus \Delta^{r-1}$ where $r$ is such that $\Delta^r = T_qM$, the complexity is always maximal, i.e., $\sigma_n(\gamma, \varepsilon) \asymp \varepsilon^{-r}$.

# Remarks

- The proof is based on estimates of the value function associated to the optimal control problem, generalizing the sub-Riemannian Ball-Box theorem [D.P., COCV, 2014].
- We studied also two other notions of complexity, where we track the curve/path by interpolation, and no metric is assumed.




- We studied also another cost

$$\mathcal{I}(u, T) = \int_0^T \sqrt{1 + \sum_{i=1}^m u_i(t)^2 \, dt}.$$

# Future work

- We would like to get rid of the assumption $f_0 \subset \Delta^s \setminus \Delta^{s-1}$, in particular focusing on the complexity of curves (or paths) $\Gamma$ s.t. $f_0|_\Gamma \equiv 0$.
- Once we have these weak asymptotic estimates it is natural to look for strong asymptotic estimates and constructive algorithms for the minimizers of the complexities (*à la* Gauthier-Zakalyukin).
- We would like also to treat the weak Hörmander assumption (i.e., that $\text{Lie}_q\{f_0, f_1, \dots, f_m\} = T_qM$) starting from systems with linear drift in $\mathbb{R}^n$, like mechanical systems. In this case quasi-static movements (i.e., the curves such that $f_0|_\Gamma \equiv 0$) are of great interest.

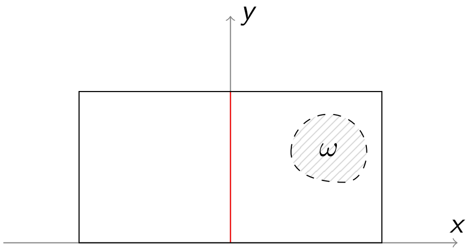
# Outline

# Grushin Plane

Consider the (non-equiregular) sub-Riemannian control system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = u_1(t)\underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{X} + u_2(t)\underbrace{\begin{pmatrix} 0 \\ x \end{pmatrix}}_{Y}, \qquad \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 .$$

Introduced in the context of hypoelliptic operators ($L = X^2 + Y^2 = \partial_x^2 + x^2\partial_y^2$) by Baouendi (1967), Grushin (1970), Franchi-Lanconelli (1984).




It is possible to control the heat evolution associated with $L$ in a square containing the singularity $\mathcal{Z} = \{x = 0\}$, by means of a control localized on one side (Beauchard, Cannarsa and Guglielmi (to appear)).

The vector fields $X$ and $Y$ are linearly independent on $\mathbb{R}^2 \setminus \mathcal{Z}$.

- They define on $\mathbb{R}^2 \setminus \mathcal{Z}$ the Riemannian metric and volume

$$\mathbf{g} = dx^2 + \frac{1}{x^2} dy^2 \qquad dV = \frac{1}{|x|} dx\, dy.$$

- The singular Laplace-Beltrami operator

$$\mathcal{L}\, u = \operatorname{div} \nabla\, u = \partial_x^2 u - \frac{1}{x} \partial_x u + x^2 \partial_y^2 u.$$

Boscain and Laurent (2013): the heat and the Schrödinger evolutions associated with this operator cannot cross the singularity. Namely,

$$\operatorname{supp} u(0) \subset \{x > 0\} \implies \operatorname{supp} u(t) \subset \{x > 0\} \text{ for any } t > 0,$$

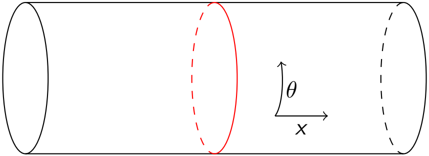
where $u$ is solution of the the heat or of the Schrödinger equations associated with $\mathcal{L}$.

**Open problem**

Is the heat absorbed or conserved by the singularity?

# Geometrical setting

Consider the manifold $M = \big(\mathbb{R} \setminus \{0\}\big) \times \mathbb{S}^1$.




With the control system

$$\frac{d}{dt}\begin{pmatrix} x \\ \theta \end{pmatrix} = u_1(t)X(x,\theta) + u_2(t)\Theta(x,\theta),$$

$$X = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \Theta = \begin{pmatrix} 0 \\ |x|^\alpha \end{pmatrix}, \qquad \alpha \in \mathbb{R}$$

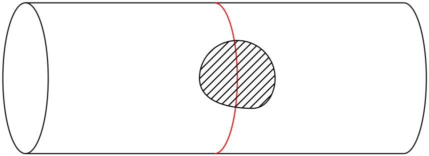
This system defines the Riemannian metric and volume:

$$\mathbf{g}_\alpha = dx^2 + |x|^{-2\alpha}\, d\theta^2, \quad dV = |x|^{-\alpha}\, dx\, dy$$

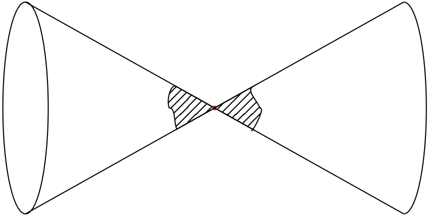
# Topological interpretation

$$M_\alpha = \mathbb{R} \times \mathbb{S}^1, \qquad X = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \Theta = \begin{pmatrix} 0 \\ |x|^\alpha \end{pmatrix}, \qquad \alpha \in \mathbb{R}\,.$$

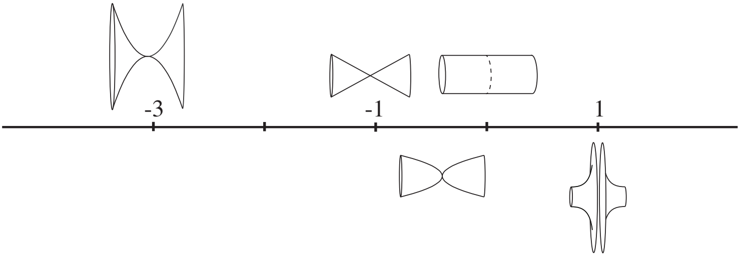
- If $\alpha \geq 0$, the topology is that of a cylinder



- If $\alpha < 0$, the topology is that of a cone.



# Metric interpretation




- $\alpha = -1$: Cone
- $\alpha = 0$: Cylinder
- $\alpha = 1$: Grushin plane compactified on one direction
- $\alpha < -1$: conical surface of revolution with profile $x^{-\alpha}$
- $\alpha \in (-1, 0)$: not embeddable in $\mathbb{R}^3$, but we can picture it as a conical surface of rotation
- $\alpha > 0$: not embeddable in $\mathbb{R}^3$, but we think of an “anti-conic” surface of revolution

# The Laplace-Beltrami operator

$$\mathcal{L}\, u = \operatorname{div} \nabla\, u = \partial_x^2 - \frac{\alpha}{x}\partial_x + |x|^{2\alpha}\partial_\theta^2.$$

- Schrödinger equation for a free particle:

$$i\frac{\partial}{\partial t}u = -\,\mathcal{L}\, u,$$

- Heat equation:

$$\frac{\partial}{\partial t}u = \mathcal{L}\, u.$$

## Questions

1. Is it possible to send a quantum particle from one side to the other of the singularity?
2. Can heat cross the singularity?
3. Is total heat conserved or the singularity absorbs it?

# The Laplace-Beltrami operator on $L^2(M, dV)$

Due to the singularity $\mathcal{Z} = \{x = 0\}$, we define $\mathcal{L}$ on $C_c^\infty(M)$.

$$\text{Give meaning to } \mathcal{L} \text{ on the singularity } \mathcal{Z}$$
$$\updownarrow$$
$$\text{Study the self-adjointness of } \mathcal{L}$$

## Classical results

In any Hilbert space the following equivalences hold,

| | | |
|---|---|---|
| $A$ self-adjoint operator | $\longleftrightarrow$ | $e^{-itA}$ strongly continuous group of unitary transformations |
| $A$ self-adjoint operator non-positive definite | $\longleftrightarrow$ | $e^{tA}$ strongly continuous semi-group |

### Definition

An operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ (we will always assume $D(A)$ to be dense in $\mathcal{H}$) is *self-adjoint* if

- $A$ is *symmetric* (i.e., if $(Au, v)_{\mathcal{H}} = (u, Av)_{\mathcal{H}}$ for any $u, v \in D(A)$),
- $D(A) = D(A^*)$.

The operator $\mathcal{L}|_{C_c^\infty(M)} : C_c^\infty(M) \to L^2(M, dV)$ is

- **Symmetric:** since if $u \in \mathrm{C}_\mathrm{c}^\infty(M)$, by integration by parts we get

$$(\mathcal{L}\, u, v)_{L^2(M,dV)} = (u, \mathcal{L}\, v)_{L^2(M,dV)} + \cancel{\left(\partial_x u\, v - u\partial_x v\right)}\Big|_{0^-}^{0^+}.$$

  for any $v \in L^2(M, dV)$ s.t. $\mathcal{L}\, v \in L^2(M, dV)$.
- **Not self-adjoint:** since

$$D(\mathcal{L}^*) = \{v \in L^2(M, dV) \mid \mathcal{L}\, v \in L^2(M, dV)\}.$$

# Self-adjoint extensions of $\mathcal{L}|_{C_c^\infty(M)}$

**Definition**

An operator $A$ is a self-adjoint extension of $\mathcal{L}|_{C_c^\infty(M)}$ if

$$D\left(\mathcal{L}|_{C_c^\infty(M)}\right) \subset D(A) = D(A^*) \subset D(\mathcal{L}^*)$$
$$Au = \mathcal{L}^* u \quad \text{for any } u \in D(A).$$

The Friedrichs extension $\mathcal{L}_F$ always exists and has domain

$$H_0^2(M, dV) = \{u \in H_0^1(M, dV) \mid \mathcal{L} u \in L^2(M, dV)\}$$

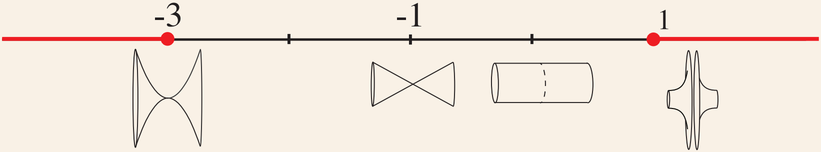
*Two possibilities:*

1. There exists only *one* self-adjoint extension $\longrightarrow \mathcal{L}|_{C_c^\infty(M)}$ is *essentially self-adjoint*.
2. There are *infinitely many* self-adjoint extensions.

# Self-adjoint extensions of $\mathcal{L}|_{C_c^\infty(M)}$

**Theorem**

*For $\alpha \notin (-3, 1)$ the operator $\mathcal{L}|_{C_c^\infty(M)}$ is essentially self-adjoint.*




When $\mathcal{L}$ is essentially self-adjoint nothing can cross the singularity.

$$D(\mathcal{L}_F) = H_0^2(M, dV) = H_0^2(\mathbb{R}_+ \times \mathbb{S}^1, dV) \oplus H_0^2(\mathbb{R}_- \times \mathbb{S}^1, dV)$$

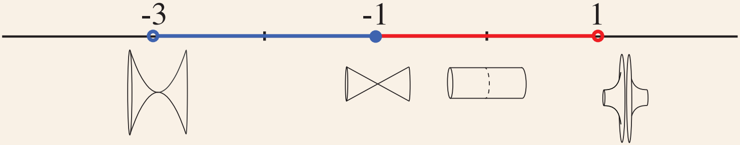
$$\implies \mathcal{L}_F = \mathcal{L}_F^+ \oplus \mathcal{L}_F^-$$

$\implies$ For $\alpha \notin (-3, 1)$ the Schrödinger evolution and the heat diffusion cannot cross the singularity.

# Free particle transmission

**Theorem**

*For $\alpha \notin (-3, 1)$ it is not possible to transmit information by means of the Schrödinger equation. On the other hand, for $\alpha \in (-3, -1]$ it is possible to transmit only the average value of the function, while for $\alpha \in (-1, 1)$ we can obtain full communication through the singularity.*




- For $\alpha \in (-1, 0)$ the manifold has the topology of a cone, but we can transmit "rotational information".
- Result obtained via a Fourier decomposition on the $\theta$ variable.

# Transmission of heat

To accept a self-adjoint extension of $\mathcal{L}|_{C_c^\infty(M)}$ as operator defining the heat evolution, we need an additional condition.

**Definition**

The self-adjoint operator $A$ on $\mathrm{L}^2(M, dV)$ is *Markovian* if it is non-positive definite and

$$u \in L^2(M, dV) \text{ s.t. } 0 \leq u \leq 1 \, a.e. \Longrightarrow 0 \leq e^{tA} u \leq 1 \, a.e.$$

- This property can be seen as a physical admissibility condition.
- Every Markov operator (with an additional regularity property, always satisfied in our cases) is the generator of a left-continuous Markov process, Fukushima (1970).

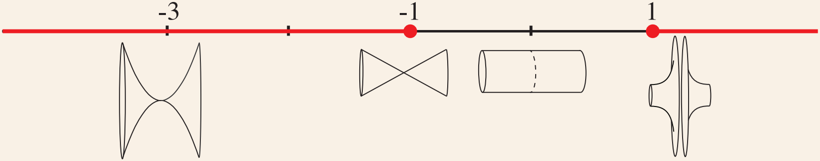
**Definition**

If the operator $\mathcal{L}|_{C_c^\infty(M)}$ admits only one Markovian extension it is *Markov unique*.

# Markov unicity and heat transmission

**Theorem**

- *If $\alpha \leq -1$ or $\alpha \geq 1$, then $\mathcal{L}|_{C_c^\infty(M)}$ is Markov unique;*
- *If $\alpha \in (-1, 1)$, then there are infinitely many Markovian extensions.*




This theorem shows that heat transmission is possible only for $\alpha \in (-1, 1)$. We let the *bridging extension* $\mathcal{L}_B$ to be the Markovian extension realizing the maximal communication:

$$D(\mathcal{L}_B) = \{u \in H^2(\bar{M}, dV) \mid u(0^+, \cdot) = u(0^-, \cdot),$$

$$\lim_{x\to 0^+} |x|^{-\alpha}\partial_x u(x, \cdot) = \lim_{x\to 0^-} |x|^{-\alpha}\partial_x u(x, \cdot) \text{ for a.e. } \theta \in \mathbb{S}^1\}$$

# The conservation of heat

The Markov property allows to extend $e^{tA}$ from $L^2(M, dV)$ to $L^\infty(M, dV)$.

## Definition

The Markov operator $A$ is *stochastically complete* if

$$e^{tA}1 = 1 \qquad \text{for any } t \geq 0.$$

## Question 3

The Markov extensions of $\mathcal{L}|_{C_c^\infty(M)}$ are stochastically complete?

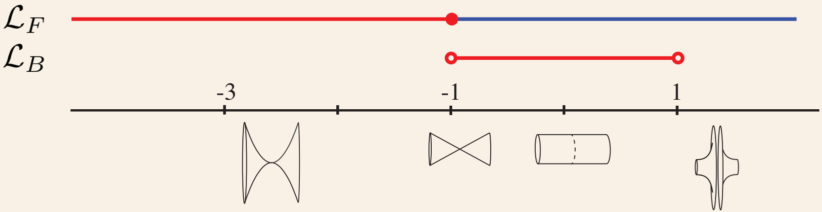
In complete Riemannian manifolds (where the Laplace-Beltrami operator is always essentially self-adjoint) this property is related with

- uniqueness of bounded solutions of the Cauchy problem (Khas'minskii 1960),
- volume explosion at infinity (Grygorian 1985).

# Stochastic completeness of $\mathcal{L}_F$ and $\mathcal{L}_B$

**Theorem**

- *The Friedrichs extension $\mathcal{L}_F$ is stochastically complete if $\alpha \leq -1$ and incomplete if $\alpha > -1$.*
- *If $\alpha \in (-1, 1)$, then the bridging extensions $\mathcal{L}_B$ is stochastically complete .*



# Remarks

- The proofs of these facts are based on potential theory and in particular on the theory of Dirichlet forms.
- The fact that $\mathcal{L}\,|_{C_c^\infty(M)}$ be Markov unique is equivalent to $H^1(M_\alpha, dV) = H_0^1(M_\alpha, dV)$.
- The stochastic completeness of an operator $A$ is equivalent to the fact that the associated left-continuous Markov process has almost surely infinite lifespan.
- We obtained more precise results on the stochastic completeness. In particular we have been able to prove that in the stochastically incomplete cases, it is actually the singularity that is absorbing the heat.

# Future work

- We would like to study the same problem from the point of view of stochastic processes, since all our results have been obtained working only on the generators.
- It would be interesting to study the scattering properties of $\mathcal{L}$ in the case $\alpha \in (-1, 1)$, using the bridging extension as a reference.
- We are working with M. Seri to extend these results to more general singular surfaces, as the 2D almost-Riemannian case (i.e., 2D structures defined by two vector fields that can become collinear but satisfy the Hörmander condition).

# Spectral analysis of the Grushin cylinder

We focus on the Laplace-Beltrami operator on one side of the singularity (where it is self-adjoint), and we call it $M_+$.

**Theorem**

*The operator $-\mathcal{L}$ on $L^2(M_+)$ has absolutely continuous spectrum $\sigma(-\mathcal{L}) = [0, +\infty)$ with embedded discrete spectrum*

$$\sigma_d(-\mathcal{L}) = \{\lambda_{n,k} = 4|k|n \mid n \in \mathbb{N}, k \in \mathbb{Z} \setminus \{0\}\}$$

## Classical question on the spectrum

- Weyl law: the asymptotic growth as $E \to +\infty$ of

$$E \mapsto N(E) = \#\{\lambda \in \sigma_\mathrm{d}(-\mathcal{L}) \mid \lambda \leq E\}.$$

# Weyl Law

## Proposition

*The Weyl law as $E \to +\infty$ of $-\mathcal{L}$ is*

$$N(E) = \frac{E}{2}\log(E) + (\gamma - 2\log 2)\frac{E}{2} + O(1),$$

*where $\gamma$ is the Euler-Mascheroni constant.*

- For a compact Riemannian manifolds $M$ of dimension 2, the leading term in the Weyl law is well known to be

$$N(E) \sim E.$$

- In some degenerate Riemannian manifolds with boundary (where the metric goes to 0 at the boundary), it is

$$N(E) \sim E \log E.$$

- In our case, the Riemannian metric explodes at the singularity.

# Remarks and perspectives

- With M. Seri we carried out a systematic study of the spectral properties of the Laplace-Beltrami operator of the Grushin cylinder and sphere (eigenfunctions, Weyl law, degeneracies).
- We also studied the Aharonov-Bohm effect in this context, i.e. the effect of hidden magnetic fluxes on the spectrum, showing that it can be surprisingly strong.

## Perspectives

- Also in the case of the Grushin sphere the Weyl law has leading term $E \log E$. We would then like to prove that this is a general fact for 2D almost-Riemannian structures of "Grushin type".
- It would be interesting to understand what happens in generic almost-Riemannian structures, where stronger singularities can appear.

Thank you for your attention.

# Ball-Box

Let $\{\partial_{z_i}\}_{i=1}^n$ be the canonical basis of $\mathbb{R}^n$ and $\mathcal{R}_{\mathbf{f_0}}(q,\varepsilon)$ the reachable set from $q$ with cost $\leq \varepsilon$. We define

$$\Xi(\eta) = \bigcup_{0\leq\xi\leq\mathcal{T}} \left(\xi\partial_{z_\ell} + \mathrm{Box}\,(\eta)\right)$$

$$\Pi(\eta) = \bigcup_{0\leq\xi\leq\mathcal{T}} \{z\in\mathbb{R}^n\colon\ |z_\ell-\xi|\leq\eta^s,\ |z_i|\leq\eta^{w_i}+\eta\xi^{\frac{w_i}{s}} \text{ pour } w_i\leq s,\ i\neq k,$$
$$\text{et } |z_i|\leq\eta(\eta+\xi^{\frac{1}{s}})^{w_i-1} \text{ pour } w_i>s\},$$

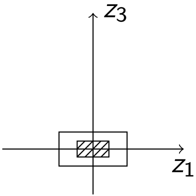
**Theorem**

*Let $z=(z_1,\ldots,z_n)$ a privileged coordinate system at $q$ for $\{f_1,\ldots,f_m\}$, rectifying $f_0$ as the $k$-th coordinate vector field $\partial_{z_\ell}$, for some $1\leq\ell\leq n$. Them, there exist $C,\varepsilon_0,T_0$ s.t., if $\mathcal{T}<T_0$, it holds*

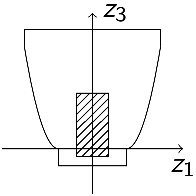
$$\Xi\left(\frac{1}{C}\varepsilon\right)\subset\mathcal{R}_{\mathbf{f_0}}(q,\varepsilon)\subset\Pi(C\varepsilon),\qquad \textit{for } \varepsilon<\varepsilon_0.$$

# Example of Ball-Box

- $f_1$ and $f_2$ control vector fields on $\mathbb{R}^3$ satisfying the Hörmander condition,
- Drift s.t. $f_0 \not\subset \Delta^1 = \text{span}\{f_1, f_2\}$.




Sub-Riemannian system.




Control-affine system.

# The Aharonov-Bohm effect on conic and anti-conic surfaces

Considering the same Aharonov-Bohm magnetic potential on the conic/anti-conic manifolds considered before yields the magnetic Laplace-Beltrami operator

$$\mathcal{L}_\alpha^b = \partial_x^2 - \frac{\alpha}{x}\partial_x + |x|^{2\alpha}\left(\partial_\theta - 2ib\partial_\theta - b^2\right)$$

Through the Fourier decomposition the operator on each $H_k$ is

$$\widehat{\mathcal{L}}_{\alpha,k}^b = \partial_x^2 - \frac{\alpha}{x}\partial_x - |x|^{2\alpha}(b-k)^2$$

## Theorem (A-B effect on self-adjointness)

*If $\alpha > -1$ the Aharonov-Bohm magnetic potential has no effect on the self-adjointness. On the other hand, if $\alpha \leq -1$, then*

1. *if $b \notin \mathbb{Z}$ the operator $\mathcal{L}_\alpha^b$ is essentially self-adjoint;*
2. *if $b \in \mathbb{Z}$ only the $k = b$ component is not essentially self-adjoint.*