Complexity in control-affine systems

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Framework

We will consider affine-control systems, i.e., systems in the form

$$\dot{q}(t) = f_0(q(t)) + \sum_{i=1}^{m} u_i(t) f_i(q(t))$$

Here,

- the point q belongs to a smooth manifold M
- the f_i's are smooth vector fields on M
- $u \in \mathrm{L}^1([0, T], \mathbb{R}^m)$

This type of system appears in many applications

- Mechanical systems
- Quantum control
- Microswimmers (Tucsnak, Alouges)
- Neuro-geometry of vision (Mumfor, Petitot)

Outline

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Motion Planning

Problem

Given $x, y \in M$, find an admissible trajectory steering the system from x to y, possibly under some constraints.

Possible constraints:

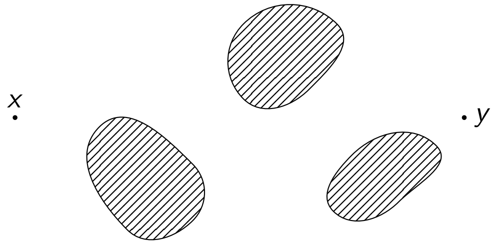
1. Avoiding some obstacles
2. Rendez-vous problem, i.e., being near certain places at certain times

Assumption

A metric with balls $B(q, \varepsilon)$ is fixed on M.

Method

Different approaches are possible. We consider the following method:



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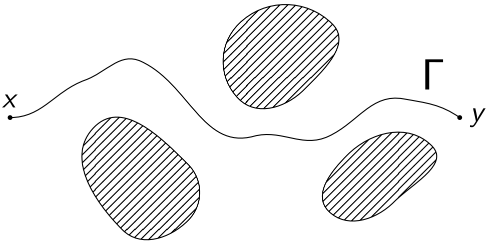
1. Find an (non-admissible) curve $\Gamma \subset M$ or a path $\gamma : [0, T] \to M$ solving the problem.



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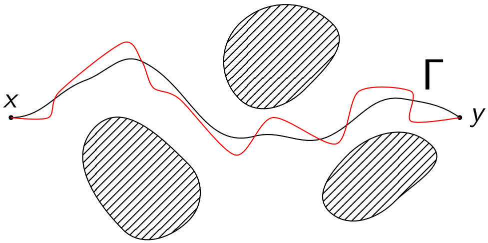
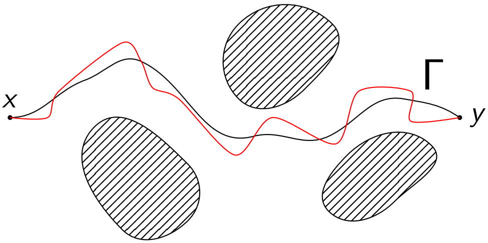
2. Track Γ or γ with an admissible trajectory.



Method

Different approaches are possible. We consider the following method:

1. Find an (non-admissible) curve $\Gamma \subset M$ or a path $\gamma : [0, T] \to M$ solving the problem. $\to$ **global topology**
2. Track Γ or γ with an admissible trajectory. $\to$ **local behavior of the control system**



We focus on quantifying the difficulty of the second step.

Outline

Complexity

Let $\mathrm{J} : \mathcal{U} \to [0, +\infty)$ be a cost function.

Definition (Complexity)

A measure of the cost of approximation of a given curve/path with a certain precision

In general:

1. we fix a set $\mathrm{Adm}(\Gamma, \varepsilon)$ of admissible controls for precision ε
2. we define complexity as

$$\sigma(\gamma, \varepsilon) = \inf_{u \in \mathrm{Adm}(\Gamma,\varepsilon)} \frac{\text{cost of } u}{\text{cost of an ``}\varepsilon \text{ piece'' of } u} = \frac{1}{\varepsilon} \inf_{u \in \mathrm{Adm}(\Gamma,\varepsilon)} \mathrm{J}(u, T).$$

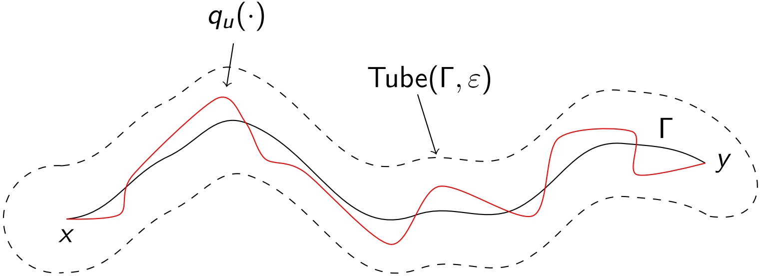
Obstacle-avoidance problem

Let $\Gamma \subset M$ be a curve, $\mathrm{Tube}(\Gamma, \varepsilon) = \bigcup_{q \in \gamma} B(q, \varepsilon)$, and

$$\mathcal{A}(\Gamma, \varepsilon) = \left\{ u \in \mathrm{L}^1([0, T], \mathbb{R}^m)) \mid \begin{array}{l} T > 0,\ q_u(T) = y, \\ q_u(\cdot) \subset \mathrm{Tube}(\Gamma, \varepsilon) \end{array} \right\}.$$

With this set we define the *tubular approximation complexity*

$$\Sigma_a(\Gamma, \varepsilon) = \frac{1}{\varepsilon} \inf_{u \in \mathcal{A}(\Gamma, \varepsilon)} \mathrm{J}(u, T).$$



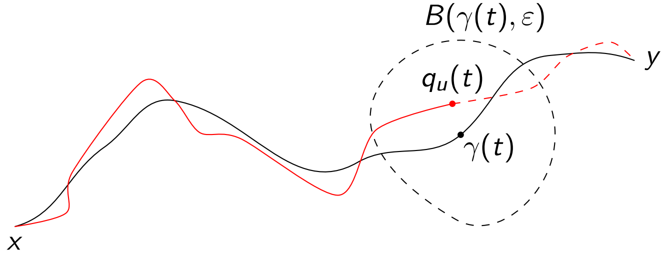
Rendez-vous problem

Let $\gamma : [0, T] \to M$ be a path and

$$\mathcal{N}(\gamma, \varepsilon) = \left\{ u \in \mathrm{L}^1([0, T], \mathbb{R}^m)) \mid \begin{array}{l} q_u(T) = y \text{ and } q_u(t) \in B(\gamma(t), \varepsilon) \\ \text{for any } t \in [0, T] \end{array} \right\}.$$

This set defines the *neighborhood approximation complexity*

$$\sigma_n(\gamma, \varepsilon) = \frac{1}{\varepsilon} \inf_{u \in \mathcal{N}(\gamma, \varepsilon)} J(u, T).$$



Outline

Particular case: sub-Riemannian control systems

A sub-Riemannian (or nonholonomic) control system is a control-affine system without drift

$$\dot{q}(t) = \sum_{i=1}^{m} u_i(t) f_i(q(t)),$$

that satisfies the Hörmander condition, i.e., such that

$$\mathrm{Lie}_q\{f_1, \dots, f_m\} = T_q M, \qquad \text{for any } q \in M.$$

1. The value function associated to this system w.r.t. the L^1 cost is a distance, called sub-Riemannian distance.
2. Due to the linearity of the system, we can always reparametrize trajectories without changing their cost. Hence,

$$\text{Tubular approximation complexity} \iff \text{Neighborhood approximation complexity}$$

Sub-Riemannian complexities

- Introduced by Gromov (1996) in a different context.

- *Weak equivalence:*

$$\sigma(\Gamma,\varepsilon) \asymp g(\varepsilon) \iff C_1 \leq \frac{\sigma(\Gamma,\varepsilon)}{g(\varepsilon)} \leq C_2 \quad \text{for } \varepsilon \downarrow 0.$$

Studied by Jean (2003).

- *Strong equivalence:*

$$\sigma(\Gamma,\varepsilon) \simeq g(\varepsilon) \iff \lim_{\varepsilon\downarrow 0} \frac{\sigma(\Gamma,\varepsilon)}{g(\varepsilon)} = 1.$$

Studied in a series of paper by Gauthier, Zakalyukin, et al.

General case

Recall the general form of a control-affine system

$$\dot{q}(t) = f_0(q(t)) + \sum_{i=1}^{m} u_i(t) f_i(q(t)).$$

We will consider:

- *strong Hörmander condition:* $\mathrm{Lie}_q\{f_1, \ldots, f_m\} = T_q M$ for any $q \in M$.
- The set of controls is
$$\mathcal{U} = \bigcup_{t \in (0,T]} \mathrm{L}^1([0,T], \mathbb{R}^m).$$
- The cost J is the L^1-norm of u.

Consequences:

1. The associated driftless system ($f_0 = 0$) is a sub-Riemannian system.
2. Small time local controllability.

Complexities for control-affine systems

- We will use the sub-Riemannian metric to define the complexities.
- Since the system is not linear, we cannot reparametrize the trajectories, and hence

$$\begin{array}{c}\text{Tubular approximation}\\ \text{complexity}\end{array} \not\Longleftrightarrow \begin{array}{c}\text{Neighborhood approximation}\\ \text{complexity}\end{array}$$

For any $q \in M$, $s \in \mathbb{N}$, let

$$\Delta^s(q) = \operatorname{span}\{[f_{i_1}, [f_{i_2}, [\dots, f_{i_k}]\dots]](q) \mid 1 \le k \le s,\ 1 \le i_j \le m\}.$$

$$\Delta^1(q) \subset \Delta^2(q) \subset \dots \subset \Delta^r(q) = T_qM$$

Hypothesis

Equiregularity: for any $s \in \mathbb{N}$, $\dim \Delta^s$ does not depend on the point $q \in M$.

Theorem

Let $f_0 \subset \Delta^s \setminus \Delta^{s-1}$.

- *Let* $\Gamma \subset M$ *be a smooth curve. Let* k *such that* $T\Gamma \subset \Delta^k$ *and* $T\Gamma \not\subset \Delta^{k-1}$. *Then, if* $\mathcal{T}$ *is sufficiently small, we have*
$$\Sigma_a(\Gamma, \varepsilon) \asymp \frac{1}{\varepsilon^k}$$
- *Let* $\gamma : [0, T] \to M$ *be a path and* k *such that* $\dot\gamma \in \Delta^k$ *and* $\dot\gamma \notin \Delta^{k-1}$. *If, moreover,* $s = k$, *we assume that* $\dot\gamma \neq f_0(\gamma) \mod \Delta^{s-1}(\gamma)$. *Then*
$$\sigma_n(\gamma, \varepsilon) \asymp \frac{1}{\varepsilon^{\max\{s,k\}}}$$

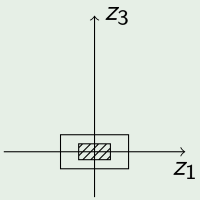
- The complexity of **curves** is not sensible to the drift.
- The complexity of **paths** depends on the drift. In particular, when $f_0 \subset \Delta^r \setminus \Delta^{r-1}$ where r is such that $\Delta^r = T_qM$, the complexity is always maximal, i.e., $\sigma_n(\gamma, \varepsilon) \asymp \varepsilon^{-r}$.

Techniques and Remarks

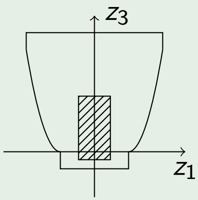
- Estimates of the shape of the reachable set for cost less than ε associated to the optimal control problem, generalizing the sub-Riemannian Ball-Box theorem.

Example

- f_1 and f_2 control vector fields on $\mathbb{R}^3$ satisfying the Hörmander condition,
- Drift s.t. $f_0 \not\subset \Delta^1 = \mathrm{span}\{f_1, f_2\}$.



Sub-Riemannian system.



Control-affine system.

Techniques and Remarks (continued)

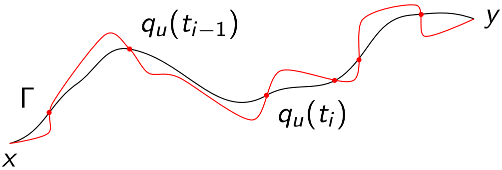
- To obtain these estimates it is necessary to reduce the control system with drift to a driftless but time-dependent system.

$$\dot{q} = f_0(q) + \sum_{i=1}^{m} u_i f_i(q) \quad \longrightarrow \quad \dot{q} = \sum_{i=1}^{m} u_i (e^{-tf_0})_* f_i(q).$$

- For this system we can define a generalization of the nilpotent approximation, that yields the estimates.

Final remarks

- We studied also two other notions of complexity, where we track the curve/path by interpolation, and no metric is assumed.



- We studied also another cost

$$\mathcal{I}(u,T) = \int_0^T \sqrt{1 + \sum_{i=1}^m u_i(t)^2\, dt}.$$

Thank you for your attention.

Ball-Box

Let $\{\partial_{z_i}\}_{i=1}^n$ be the canonical basis of $\mathbb{R}^n$ and $\mathcal{R}_{\mathbf{f_0}}(q,\varepsilon)$ the reachable set from q with cost $\leq \varepsilon$. We define

$$\Xi(\eta) = \bigcup_{0\leq\xi\leq\mathcal{T}} \left(\xi\partial_{z_\ell} + \mathrm{Box}\,(\eta)\right)$$

$$\Pi(\eta) = \bigcup_{0\leq\xi\leq\mathcal{T}} \{z \in \mathbb{R}^n \colon\ |z_\ell - \xi| \leq \eta^s,\ |z_i| \leq \eta^{w_i} + \eta\xi^{\frac{w_i}{s}} \text{ pour } w_i \leq s,\ i \neq k,$$

$$\text{et } |z_i| \leq \eta(\eta + \xi^{\frac{1}{s}})^{w_i - 1} \text{ pour } w_i > s\},$$

Theorem

Let $z = (z_1, \dots, z_n)$ a privileged coordinate system at q for $\{f_1, \dots, f_m\}$, rectifying f_0 as the k-th coordinate vector field ∂_{z_ℓ}, for some $1 \leq \ell \leq n$. Them, there exist C, ε_0, T_0 s.t., if $\mathcal{T} < T_0$, it holds

$$\Xi\left(\frac{1}{C}\varepsilon\right) \subset \mathcal{R}_{\mathbf{f_0}}(q,\varepsilon) \subset \Pi(C\varepsilon), \qquad \textit{for } \varepsilon < \varepsilon_0.$$