Exercises

November 11, 2013

Exercise 1. Given $f, g \in \chi(M)$ and $q \in M$, consider the curve

$$\gamma(t) = e^{-tg} \circ e^{-tf} \circ e^{tg} \circ e^{tf}(q),$$

where e^{tf} and e^{tg} are the flows at time t of the vector fields f and g, respectively. Prove that

$$\left.\frac{d}{dt}\right|_{t=0} \gamma(\sqrt{t}) = [f, g](q).$$

Exercise 2. Consider a control system of the form

$$\dot{q} = \sum_{i=1}^{m} u_i f_i(q), \qquad q \in M,\ f_i \in \chi(M),\ u_i \in \mathbb{R}\,.$$

Prove that such system is controllable for the following choices of $f_1, \dots, f_m$ and M.

1. $M = \mathbb{R}^2$ with coordinates (x_1, x_2):
   (a) *Grushin.* $f_1 = (1, 0),\ f_2 = (0, x_1)$.
2. $M = \mathbb{R}^3$ with coordinates (x_1, x_2, x_3):
   (a) *Heisenberg.* $f_1 = (1, 0, 0),\ f_2 = (0, 1, x_1)$.
   (b) *Martinet.* $f_1 = (1, 0, \frac{1}{2}x_2^2),\ f_2 = (0, 1, 0)$
3. $M = \mathbb{R}^4$ with coordinates (x_1, x_2, x_3, x_4):
   (a) *Engel.* $f_1 = (1, 0, 0, 0),\ f_2 = (0, 1, x_1, x_1x_2)$.
   (b) $f_1 = (1, 0, 0, 0),\ f_2 = (0, 1, 0, x_1),\ f_3 = (0, 0, 1, 0)$.
4. $M = \mathbb{R}^5$ with coordinates $(x_1, x_2, x_3, x_4, x_5)$:
   (a) *Cartan.* $f_1 = (1, 0, 0, 0, 0),\ f_2 = (0, 1, x_1, \frac{1}{2}x_1^2, x_1x_2)$.
   (b) *Goursat rank 2.* $f_1 = (1, 0, 0, 0, 0),\ f_2 = (0, 1, x_1, \frac{1}{2}x_1^2, \frac{1}{6}x_1^3)$.
   (c) $f_1 = (1, 0, 0, -\frac{1}{2}x_2, 0),\ f_2 = (0, 1, 0, \frac{1}{2}x_1, -\frac{1}{2}x_3),\ f_3 = (0, 0, 1, 0, \frac{1}{2}x_2)$.

(d) *Goursat rank 3.* $f_1 = (1, 0, 0, -\frac{1}{2}x_2, -\frac{1}{3}x_1 x_2)$, $f_2 = (0, 1, 0, \frac{1}{2}x_1, \frac{1}{3}x_1^2)$, $f_3 = (0, 0, 1, 0, 0)$.

(e) *Bi-Heisenberg.* $f_1 = (1, 0, 0, 0, -\frac{1}{2}x_2)$, $f_2 = (0, 1, 0, \frac{1}{2}x_1, 0)$, $f_3 = (0, 0, 1, 0, -\frac{1}{2}x_4)$, $f_4 = (0, 0, 0, 1, \frac{1}{2}x_3)$.

Exercise 3. Consider the following system on $\mathbb{R}^2$ with coordinates (x, y),

$$\frac{d}{dt}(x, y) = (u_1, f(x)u_2), \qquad u_1, u_2 \in \mathbb{R}\,.$$

Here $f \in C^\infty(\mathbb{R})$ is such that $\text{supp} f \subset [0, \infty)$. Prove that:

1. The system is not Lie-bracket generating.
2. Nevertheless, it is controllable.

Exercise 4. Consider the following system on $\mathbb{R}^2$ with coordinates (x, y),

$$\frac{d}{dt}(x, y) = (u_1, u_2|x|^\alpha), \qquad u_1, u_2 \in \mathbb{R},\ \alpha \in \mathbb{R}\,.$$

1. For which values of the parameter α the corresponding family of vector fields is Lie-bracket generating?
2. For which values of the parameter the system is controllable?

Exercise 5. For which values of the parameter $\alpha \in \mathbb{R}$ the following system on $\mathbb{R}^2$ with coordinates (x, y) is controllable?

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2 & \alpha - 3 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ \alpha^2 - \alpha & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \qquad u,\, v \in \mathbb{R}\,.$$

Exercise 6. Study the controllability of the following control system on $\mathbb{R}^3$ with coordinates (x, y, z),

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} y \\ -x \\ 0 \end{pmatrix} + u_1 \begin{pmatrix} 0 \\ z \\ 0 \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad u_1, u_2 \in \mathbb{R}\,.$$

Exercise 7. Consider the control system on $\mathbb{R}^2$ with coordinates (x, y)

$$\begin{cases} \dot{x} = 1 \\ \dot{y} = -u\ x \end{cases}, \qquad u \in \mathbb{R}\,.$$

Is it controllable? What is the set of points that can be reached starting from the origin?