A semidiscrete version of the Petitot model as a plausible model for anthropomorphic image reconstruction and pattern recognition

D. Prandi and J.-P. Gauthier

Contents

Part I. Preliminary results

1. Prerequisites

1.1. Harmonic analysis on locally compact abelian groups

Let $\mathbb{G}$ be a locally compact abelian group. A *character* of $\mathbb{G}$ is a continuous group homomorphism $\lambda : \mathbb{G} \to \mathbb{C}$, which implies that $|\lambda(a)| = 1$ for any $a \in \mathbb{G}$. Defining the product of two characters as the pointwise multiplication, the inverse as the complex conjugation, the set

$$\hat{\mathbb{G}} = \{\lambda \mid \lambda \text{ is a charachter of } \mathbb{G}\},$$

endowed with the topology of uniform convergence on compact sets is a locally compact abelian group, called the *(Pontryagin) dual group* of $\mathbb{G}$.

Let $\Omega : \mathbb{G} \to \widehat{\widehat{\mathbb{G}}}$ be defined by

$$\Omega_x(\lambda) := \lambda(x).$$

This is a continuous group homomorphism, and hence $\mathbb{G} \subset \widehat{\widehat{\mathbb{G}}}$.

Theorem 1.1.1 (Pontryagin duality)**.** *The map Ω is a group isomorphism, and thus $\mathbb{G}$ is canonically isomorphic to the dual of $\widehat{\mathbb{G}}$.*

The Fourier transform allows to carry this isomorphism to the level of complex-valued functions defined on $\mathbb{G}$ and $\widehat{\mathbb{G}}$. Namely, endow $\mathbb{G}$ with its Haar measure and for any $f \in L^2(\mathbb{G}) \cap L^1(\mathbb{G})$ define its Fourier transform $\hat{f} \in L^2(\widehat{\mathbb{G}})$ by

$$\hat{f}(\lambda) := \int_{\mathbb{G}} f(x)\bar{\lambda}(x)\, dx.$$

Observe, in particular, that letting $\operatorname{avg} f = \int_{\mathbb{G}} f(x)\, dx$ it holds $\operatorname{avg} f = \hat{f}(\hat{o})$, where $\hat{o}(x) = 1$ is the identity of $\widehat{\mathbb{G}}$. We have the following.

Theorem 1.1.2 (Plancherel Theorem)**.** *There exists a unique measure $d\lambda$ on $\widehat{\mathbb{G}}$, called* Plancherel measure*, such that the above defined Fourier transform can be extended to an isometry $\mathcal{F} : L^2(\mathbb{G}) \to L^2(\widehat{\mathbb{G}})$. In particular, whenever $f \in L^2(\mathbb{G}) \cap L^1(\mathbb{G})$ and $\hat{f} \in L^2(\widehat{\mathbb{G}}) \cap L^1(\widehat{\mathbb{G}})$ it holds that*

$$\mathcal{F}^{-1}(\hat{f})(x) = \int_{\widehat{\mathbb{G}}} \hat{f}(\lambda)\lambda(x)\, d\lambda.$$

Remark 1.1.3. *When $\mathbb{G} = \mathbb{R}$ the above procedure yields the classical Fourier transform. Indeed, the Haar measure of $\mathbb{R}$ is the Lebesgue measure, $\widehat{\mathbb{R}} \cong \mathbb{R}$ can be realized as the set of $x \mapsto e^{2\pi i \lambda x}$ for $\lambda \in \mathbb{R}$ and the Plancherel measure is again the normalized Lebesgue measure.*

Letting the left regular representation of $\mathbb{G}$, $x \mapsto \tau_x \in \mathcal{U}(L^2(\mathbb{G}))$, be defined as $\tau_x f(y) = f(y - x)$, the fundamental property of the Fourier transform for our purposes is the following.

Theorem 1.1.4. *For any $f, g \in L^2(\mathbb{G})$ and any $x \in \mathbb{G}$ it holds that*

$$f = \tau_x g \iff \hat{f}(\lambda) = \bar{\lambda}(x)\, \hat{g}(\lambda) \quad \forall \lambda \in \widehat{\mathbb{G}}.$$

1.2. Fourier transform on locally compact non-commutative groups

Let $\mathbb{G}$ be a locally compact unimodular group. A *unitary representation* T of $\mathbb{G}$ is a continuous[1] homomorphism $T : \mathbb{G} \to \mathcal{U}(\mathcal{H}_T)$, where $\mathcal{H}_T$ is a complex (possibly infinite dimensional) Hilbert space. A representation T is *irreducible* if there is no nontrivial closed subspace of $\mathcal{H}_T$ which is invariant for all the $T(a)$, $a \in \mathbb{G}$. Two representations T, T' are *equivalent* if there is a linear invertible operator $A : \mathcal{H}_T \to \mathcal{H}_{T'}$ such that $A \circ T = T' \circ A$. In this case we write $T \cong T'$.

The *dual set of* $\mathbb{G}$ is the set $\widehat{\mathbb{G}}$ of all equivalence classes of unitary irreducible representations of $\mathbb{G}$. We remark that $\widehat{\mathbb{G}}$ has in general no group structure. However, this is enough to generalize the Fourier transform to this context. Let $f \in L^2(\mathbb{G}) \cap L^1(\mathbb{G})$, then its Fourier transform is defined by

$$\hat{f}(T) = \int_{\mathbb{G}} f(a)\, T(a)^{-1}\, da, \qquad \forall T \in \widehat{\mathbb{G}}. \tag{1}$$

Observe that $\hat{f}(T)$ is an Hilbert-Schmidt operator on $\mathcal{H}_T$. We have the following generalization of Theorem 1.1.2.

Theorem 1.2.1 (Unimodular non-commutative Plancherel Theorem)**.** *Let $\mathbb{G}$ be a locally compact unimodular group. Then, there exists a (unique) Plancherel measure $\hat{\mu}_{\mathbb{G}}$ on $\widehat{\mathbb{G}}$ such that the above definition can be extended to an isometry $\mathcal{F} : L^2(\mathbb{G}) \to L^2(\widehat{\mathbb{G}}, \hat{\mu}_{\mathbb{G}})$. In particular, the following inversion formula holds*

$$f(a) = \int_{\widehat{\mathbb{G}}} \mathrm{Tr}\left(\hat{f}(T) \circ T(a)\right) d\hat{\mu}_{\mathbb{G}}(T).$$

More generally, if T is a unitary representation of $\mathbb{G}$ – not necessarily irreducible – one can define the Fourier transform $\hat{f}(T)$ by the same formula (1).

As in the abelian case, the Fourier transform has a nice behavior w.r.t. to the action of the left regular representation $\Lambda : \mathbb{G} \mapsto \mathcal{U}(L^2(\mathbb{G}))$, defined by $\Lambda(a)f(b) := f(a^{-1}b)$.

Theorem 1.2.2 (Fundamental property w.r.t. the action of left regular representation)**.** *For any $f, g \in L^2(\mathbb{G})$ and any $a \in \mathbb{G}$ it holds*

$$f = \Lambda(a)g \iff \hat{f}(T) = g(T) \circ T^{-1}(a) \quad \forall T \in \widehat{\mathbb{G}}.$$

1.3. Chu Duality

Chu duality is an extension of the dualities of Pontryagin (see Theorem 1.1.1) and Tannaka (for compact groups) to certain more general groups. In particular, it applies to *Moore groups*, i.e., those groups whose unitary irreducible representations are all finite dimensional. Here the difficulty is to find a suitable notion of bidual, carrying a group structure. See [6].

[1] With respect to the strong topology of $\mathcal{U}(\mathcal{H}_T)$

Let $\mathrm{Rep}_n(\mathbb{G})$ denote the set of continuous unitary representations of $\mathbb{G}$ over $\mathbb{C}^n$. Taking as a basis of neighborhoods at $T \in \mathrm{Rep}_n(\mathbb{G})$ the sets

$$W(T, K, \varepsilon) := \{\rho \in \mathrm{Rep}_n(\mathbb{G}) \mid \|T(a) - \rho(a)\|_{\mathrm{HS}} \leq \varepsilon \quad \forall a \in K\}, \quad \forall \epsilon > 0, K \subset \mathbb{G} \text{ compact},$$

the set $\mathrm{Rep}_n(\mathbb{G})$ is a topological space which turns out to be locally compact since $\mathbb{G}$ is so. The *Chu dual* of $\mathbb{G}$ is the topological sum

$$\mathrm{Rep}(\mathbb{G}) := \bigcup_{n \geqslant 1} \mathrm{Rep}_n(\mathbb{G}).$$

A *quasi-representation* of $\mathbb{G}$ is a continuous map Q from $\mathrm{Rep}(\mathbb{G})$ to $\bigcup_{n \geqslant 1} \mathcal{U}(\mathbb{C}^n)$ such that for any $T \in \mathrm{Rep}_{n(T)}(\mathbb{G})$, $T' \in \mathrm{Rep}_{n(T')}(\mathbb{G})$, and any $U \in \mathcal{U}(\mathbb{C}^{n(T)})$ it holds

1. $Q(T) \in \mathcal{U}(\mathbb{C}^{n(T)})$;
2. $Q(T \oplus T') = Q(T) \oplus Q(T')$;
3. $Q(T \otimes T') = Q(T) \otimes Q(T')$;
4. $Q(U \circ T \circ U^{-1}) = U \circ Q(T) \circ U^{-1}$;

The set of quasi-representations of $\mathbb{G}$ is denoted by $\mathrm{Rep}(\mathbb{G})^\vee$ and is called the *Chu quasi-dual*. Setting $E(T) := \mathrm{Id}_{n(T)}$ and $Q^{-1}(T) = Q(T^{-1})$, the Chu quasi-dual is an Hausdorff topological group with identity E. Finally, we can define the continuous group homomorphism $\Omega : \mathbb{G} \mapsto \mathrm{Rep}(\mathbb{G})^\vee$ as

$$\Omega_a(T) := T(a).$$

Definition 1.3.1. *A locally compact group* $\mathbb{G}$ *has the* Chu duality property *if* Ω *is a topological group isomorphism.*

The main result is then the following.

Theorem 1.3.2 (Chu duality)**.** *Whenever* $\mathbb{G}$ *is abelian or Moore it has the Chu duality property.*

Observe that, since any compact group is Moore, the above theorem shows that Chu duality contains Tannaka duality.

2. General setting

In most of the paper we will consider the general setting of a semidirect product $\mathbb{G} = \mathbb{H} \rtimes \mathbb{K}$, where

- $\mathbb{H}$ is an abelian separable connected locally compact group.
- $\mathbb{K}$ is an abelian finite group of cardinality N.

- The action $k \in \mathbb{K} \mapsto R_k \in \mathrm{Aut}(\mathbb{H})$ of $\mathbb{K}$ on $\mathbb{H}$ is free and the Haar measure of $\mathbb{H}$ is invariant under the R_k's.

The above assumptions guarantee that $\mathbb{G}$ is unimodular [7, Ch. II, Prop. 28]. Later on we will explicitly compute the unitary irreducible representations of $\mathbb{G}$, which will be finite dimensional, thus proving that $\mathbb{G}$ is a Moore group.

Remark 2.0.3. *The assumption of the action of $\mathbb{K}$ to be free could probably be removed. However, this would yield to a more complicated description of the representations of $\mathbb{G}$ and is outside the scope of this work, whose main motivation is $SE(2,N) = \mathbb{R}^2 \rtimes \mathbb{Z}_N$.*

Additive notation is used for both $\mathbb{H}$ and $\mathbb{K}$. We denote the identity of $\mathbb{H}$ by o and that of $\mathbb{K}$ by e. The letters x, y, z are reserved for elements of $\mathbb{H}$, while $k, h, \ell, \alpha, \beta$ are elements of $\mathbb{K}$. Elements of the Pontryagin duals $\widehat{\mathbb{H}}$ and $\widehat{\mathbb{K}}$ are denoted, respectively, as $\lambda, \mu \in \widehat{\mathbb{H}}$ and $\hat{k}, \hat{h}, \ldots \in \widehat{\mathbb{H}}$. The identities of the Pontryiagin duals are $\hat{o}$ and $\hat{e}$. Elements of $\mathbb{G}$ are denoted either by $a, b \in \mathbb{G}$ or as couples $(x, k), (y, h)$.

The action of $\mathbb{K}$ on $\mathbb{H}$ induces an action of $\mathbb{K}$ on $\widehat{\mathbb{H}}$, still denoted $k \mapsto R_k$ and defined by $R_k\lambda(x) = \lambda(R_{-k}x)$. The left regular representations of $\mathbb{H}$ and $\mathbb{K}$ are called *translation* and *shift* operators and denoted by $x \mapsto \tau_x \in \mathcal{U}(L^2(\mathbb{H}))$ and $k \mapsto S^k \in \mathcal{U}(L^2(\mathbb{K}))$, respectively. Their action on $f \in L^2(\mathbb{H})$ and $v \in L^2(\mathbb{K})$ is given by

$$\tau_x f(y) \coloneqq f(y - x) \quad \text{and} \quad (S^k v)_j = v_{j-k}.$$

Clearly, when $\mathbb{K}$ is cyclic, the shift operator is completely determined by $S = S^e$ via $S^k v = S \circ \cdots \circ Sv$.

The left regular representation of $\mathbb{G}$ is denoted by $\Lambda : \mathbb{G} \to \mathcal{U}(L^2(\mathbb{G}))$, and its action on $f \in L^2(\mathbb{G})$ is $\Lambda(a)f(b) = f(a^{-1}b)$. Exploiting the semidirect product structure of $\mathbb{G}$ we can consider the quasi-regular representation of $\mathbb{G}$, denoted by $\pi : \mathbb{G} \to \mathcal{U}(L^2(\mathbb{H}))$ whose action on $f \in L^2(\mathbb{H})$ is $\pi(x,k)f(y) = f(R_{-k}(y - x))$.

The invariant closed subspaces $\mathcal{A} \subset L^2(\mathbb{H})$ for π are of the form

$$\mathcal{A} = \mathcal{A}_U = \{f \in L^2(\mathbb{H}) \mid \operatorname{supp} \hat{f} \subset U\}, \tag{2}$$

where $U \subset \widehat{\mathbb{H}}$ is a measurable, $\mathbb{K}$ invariant subspace.

We will also consider the representation $\hat{\pi}$ obtained by conjugating π with the Fourier transform on $\mathbb{H}$, which is readily seen to operate on $L^2(\widehat{\mathbb{H}})$ via

$$\hat{\pi}(x,k)\hat{f}(\lambda) = \mathcal{F}(\pi(x,k)f)(\lambda) = \lambda(-x)\hat{f}(R_{-k}\lambda).$$

Throughout the paper we will be interested in quotient out the effect of the action of $\mathbb{H}$ or one of its subsets on $L^2_{\mathbb{R}}(\mathbb{H})$.

Definition 2.0.4. *Let $\mathcal{A} \subset L^2(\mathbb{H})$ be invariant under the action of π and let $U \subset \mathbb{H}$. A* centering of $\mathcal{A}$ w.r.t. U *is an operator $\Phi : \mathcal{A} \to \mathcal{A}$ that acts by $\Phi(f) = \tau_{c(f)} f$, where $c : \mathcal{A} \to \mathbb{H}$ is such that, for any $f, g \in \mathcal{A}$,*

$$\Phi(f) = \Phi(g) \iff \exists x \in U \text{ s.t. } f = \tau_x g.$$

Observe that the above implies that for any $k \in \mathbb{K}$ it holds

$$\Phi(f) = R_k \Phi(g) \iff \exists x \in U \text{ s.t. } f = \pi(x,k)g.$$

It is then clear that $\Phi(\mathcal{A}) \subset \mathcal{A}$ is invariant under the action of $\mathbb{K}$.

A complete description of the unitary irreducible representations of $\mathbb{G}$ can be obtained via Mackey machinery. We recall it in the following.

Theorem 2.0.5 (Representations of semidirect products)**.** *To any $\hat{k} \in \widehat{\mathbb{K}}$ corresponds the unitary representation of $\mathbb{G}$ defined by $T^{\hat{o}\times\hat{k}} = \hat{k}$ and acting on $\mathbb{C}$. On the other hand, to any $\lambda \in \widehat{\mathbb{H}} \setminus \{\hat{o}\}$ corresponds the unitary representation T^λ acting on $L^2(\mathbb{K})$ and defined by*

$$T^\lambda(x,k) = \operatorname{diag}_h(\lambda(R_h x))\, S^k.$$

Moreover, the dual set $\widehat{\mathbb{G}}$ is the union of the set of the nontrivial orbits in $\widehat{\mathbb{H}}$ under the action of $\mathbb{K}$ and of $\{\hat{o}\} \times \widehat{\mathbb{K}}$. Indeed, for any $\ell \in \mathbb{K}$ it holds that $T^{R_\ell \lambda} \circ S^\ell = S^\ell \circ T^\lambda$ and hence T^{λ_1} is equivalent to T^{λ_2} whenever λ_1, λ_2 belongs to the same orbit. Finally, the Plancherel measure $\hat{\mu}_{\mathbb{G}}$ is supported outside of $\{\hat{o}\} \times \widehat{\mathbb{K}}$.

Remark 2.0.6. *When $\mathbb{G} = SE(2,N)$, the set of nontrivial orbits can be identified with the "slice of camembert" $\mathcal{S} \subset \widehat{\mathbb{R}^2} \cong \mathbb{R}^2$, which in polar coordinates is the set of $(\rho,\theta) \in \mathbb{R}^+_* \times [0, 2\pi/N)$.*

Given $f \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$ we let $\operatorname{avg}_f \in L^2(\mathbb{K})$ be defined as $\operatorname{avg}_f(k) \coloneqq \operatorname{avg} f(\cdot, k)$.

Proposition 2.0.7. *Let $f \in L^2(\mathbb{G})$. Then, for any $\lambda \in \widehat{\mathbb{H}} \setminus \{\hat{o}\}$ it holds that*

$$\hat{f}(T^\lambda)v(h) = \int_{\mathbb{K}} \mathcal{F}(f(\cdot, h-k))(R_{-k}\lambda)\, v(k)\, dk \qquad \forall v \in L^2(\mathbb{K}).$$

Moreover, for any $f \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$ and for any $\hat{k} \in \widehat{\mathbb{K}}$ it holds

$$\hat{f}(T^{\hat{o}\times\hat{k}}) = \widehat{\operatorname{avg}_f}(\hat{k}).$$

Proof. By Theorem 2.0.5, straightforward computations yield

$$\hat{f}(T^\lambda)v(h) = \int_{\mathbb{K}}\int_{\mathbb{H}} f(x,\ell)\bar{\lambda}(R_{h-\ell}x)v(h-\ell)\, dx d\ell = \int_{\mathbb{K}}\int_{\mathbb{H}} f(x, h-k)R_{-k}\bar{\lambda}(x)v(k)\, dx dk,$$

which implies the first statement. On the other hand, to prove the second statement it suffices to compute

$$\hat{f}(T^{\hat{o}\times\hat{k}}) = \int_{\mathbb{K}}\int_{\mathbb{H}} f(x,\ell)\hat{k}(-\ell)\, dx d\ell = \int_{\mathbb{K}} \operatorname{avg}_f(\ell)\, \overline{\hat{k}}(\ell)\, d\ell = \widehat{\operatorname{avg}_f}(\hat{k}).$$

□

2.1. Induction-Reduction theorem

Theorem 2.1.1 (Induction-Reduction Theorem)**.** *For any $\lambda_1, \lambda_2 \in \mathbb{H} \setminus \{\hat{o}\}$ it holds*

$$T^{\lambda_1} \otimes T^{\lambda_2} \cong \bigoplus_{k \in \mathbb{K}} T^{\lambda_1 + R_k \lambda_2}.$$

The equivalence $A : L^2(\mathbb{K} \times \mathbb{K}) \to \bigoplus_{k \in \mathbb{K}} L^2(\mathbb{K})$ in Theorem 2.1.1 can be explicitly computed as

$$(AV)_k(h) = A_k V(h) = V(h, h-k), \qquad \forall V \in L^2(\mathbb{K} \times \mathbb{K}). \tag{3}$$

Here, $A_k : \bigoplus_{k \in \mathbb{K}} L^2(\mathbb{K}) \to L^2(\mathbb{K})$ is the composition of A with the projection π_k on the k-th component. The inverse of A is its adjoint $A^* : \bigoplus_{k \in \mathbb{K}} L^2(\mathbb{K}) \to L^2(\mathbb{K} \times \mathbb{K})$, given by

$$(A^*\psi)(i,j) = \psi_{i-j}(i) \qquad \forall \psi = (\psi_h)_{h \in \mathbb{K}} \in \bigoplus_{h \in \mathbb{K}} L^2(\mathbb{K}).$$

Let $p_k : L^2(\mathbb{K}) \to \bigoplus_{h \in \mathbb{K}} L^2(\mathbb{K})$ be the right-inverse of π_k, defined by

$$(p_k v)_h = \begin{cases} v & \text{if } k = h, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, the action of a linear operator $\mathcal{B} : \bigoplus_{k \in \mathbb{K}} L^2(\mathbb{K}) \to \bigoplus_{k \in \mathbb{K}} L^2(\mathbb{K})$ is given by

$$(\mathcal{B}\psi)_k = \sum_{\ell \in \mathbb{K}} \mathcal{B}_{k,\ell} \psi_\ell \qquad \forall \psi = (\psi_h)_{h \in \mathbb{K}} \in \bigoplus_{h \in \mathbb{K}} L^2(\mathbb{K}),$$

where $\mathcal{B}_{k,\ell} = \pi_k \circ \mathcal{B} \circ p_\ell$ are the block components of $\mathcal{B}$.

In the following we collect some useful facts.

Proposition 2.1.2. *Let A be the equivalence in* (3)*. Then, the following hold*

- *For any linear operator $\mathcal{T} : L^2(\mathbb{K} \times \mathbb{K}) \to L^2(\mathbb{K} \times \mathbb{K})$ with components $\mathcal{T} = (\mathcal{T}_{i,j,r,s})_{i,j,r,s}$, the operator $A \circ \mathcal{T} \circ A^*$ has k, ℓ block component:*

  $$(A \circ \mathcal{T} \circ A^*)_{k,l} = A_k \circ \mathcal{T} \circ A^* \circ p_\ell = (\mathcal{T}_{i,i-k,j,j-\ell})_{i,j \in \mathbb{K}}.$$

  In particular, for a couple of linear operators $B, C : L^2(\mathbb{K}) \to L^2(\mathbb{K})$ it holds

  $$(A \circ (B \otimes C) \circ A^*)_{k,l} = (B_{i,j} C_{i-k,j-\ell})_{i,j \in \mathbb{K}}.$$

- *Let $\tilde{S}^j : \bigoplus_{k \in \mathbb{K}} L^2(\mathbb{K}) \to \bigoplus_{k \in \mathbb{K}} L^2(\mathbb{K})$ be defined by $(S^j \psi)_\ell = \psi_{\ell - j}$ for any $\psi = (\psi_h)_{h \in \mathbb{K}}$ and $\ell \in \mathbb{K}$. Then,*

  $$A \circ S^i \otimes S^j \circ A^* = \tilde{S}^{i-j} \circ \bigoplus_{k \in \mathbb{K}} S^i.$$

3. Weakly cyclic functions

A vector $v \in L^2(\mathbb{K})$ is *cyclic* if $\{S^k v\}_{k\in\mathbb{K}}$ is a basis for $L^2(\mathbb{K})$. If $\mathbb{K}$ is cyclic and finite with N elements, this is equivalent to the following circulant operator being invertible

$$\operatorname{Circ} v = \Big(v,\, Sv,\, \ldots,\, S^{N-1}v\Big).$$

Fix $f \in L^2(\mathbb{H})$. For $\lambda \in \widehat{\mathbb{H}}$ we let the vector $\omega_f(\lambda) \in L^2(\mathbb{K})$ to be

$$\omega_f(\lambda)_k = \hat{f}(R_{-k}\lambda) \qquad \forall k \in \mathbb{K}. \tag{4}$$

Here we denoted by $\hat{f}$ the abelian Fourier transform on $\mathbb{H}$. Observe that $S^k\omega_f(\lambda) = \omega_f(R_k\lambda)$.

Since $\omega_f(o) = (\hat{f}(o), \ldots, \hat{f}(o))$, the vector $\omega_f(\lambda)$ cannot be cyclic for every $\lambda \in \widehat{\mathbb{H}}$, thus motivating the following definition.

Definition 3.0.3. *A function $f \in L^2(\mathbb{H})$ is* weakly cyclic *if $\omega_f(\lambda)$ is cyclic for a.e. $\lambda \in \widehat{\mathbb{H}}$. We denote by $\mathcal{C} \subset L^2(\mathbb{H})$ the set of weakly cyclic functions.*

3.1. Real valued functions

Our arguments in the following are heavily based on exploiting the weak-cyclicity property. However, for real valued functions this turns out to be impossible in general.

Definition 3.1.1. *The action of $\mathbb{K}$ on $\mathbb{H}$ is* even *if there exists $k_0 \in \mathbb{K}$ such that $R_{k_0} = -\operatorname{Id}$.*

A necessary condition for the action to be even is that $k_0 = -k_0$. The example to keep in mind is that of the natural action of $\mathbb{Z}_N$ on $\mathbb{R}^2$ when N is even.

Proposition 3.1.2. *Let $\mathbb{K}$ be acting evenly on $\mathbb{H}$ and $f \in L^2_{\mathbb{R}}(\mathbb{H})$. Then $\omega_f(\lambda) \in \mathcal{X}$ for any $\lambda \in \widehat{\mathbb{H}}$, where $\mathcal{X}$ is the proper $\mathbb{R}$-linear subspace of $L^2(\mathbb{K})$ defined by*

$$\mathcal{X} = \Big\{ v \in L^2(\mathbb{K}) \mid v(h) = \overline{v(h+k_0)} \quad \forall h \in \mathbb{K} \Big\}.$$

In particular, $\omega_f(\lambda)$ is never cyclic.

Proof. From the parity of the action it follows that $R_{-h}\lambda = -R_{-h-k_0}\lambda$ for any $\lambda \in \widehat{\mathbb{H}}$ and $h \in \mathbb{K}$. Since $\hat{f}(\lambda) = \overline{\hat{f}(-\lambda)}$, this implies that

$$\omega_f(\lambda)_h = \hat{f}(R_{-h}\lambda) = \hat{f}(-R_{-h-k_0}\lambda) = \overline{\hat{f}(R_{-h-k_0}\lambda)} = \overline{\omega_f(\lambda)_{h+k_0}}, \qquad \forall h \in \mathbb{K},$$

which proves the statement. □

Observe that $\mathcal{X}$ is invariant under the action of the shift operator. We then say that $w \in \mathcal{X}$ is $\mathbb{R}$-*cyclic* if $\operatorname{span}\{S^k w\}_{k\in\mathbb{K}} = \mathcal{X}$, and pose the following.

Definition 3.1.3. *If the action of* $\mathbb{K}$ *on* $\mathbb{H}$ *is even, a real valued function* $f \in L^2_{\mathbb{R}}(\mathbb{H})$ *is* weakly $\mathbb{R}$-cyclic *if* $\omega_f(\lambda)$ *is* $\mathbb{R}$*-cyclic for a.e.* $\lambda \in \widehat{\mathbb{H}}$*. On the other hand, if* $\mathbb{K}$ *is not acting evenly on* $\mathbb{H}$*,* $f \in L^2_{\mathbb{R}}(\mathbb{H})$ *is weakly* $\mathbb{R}$*-cyclic if and only if it is weakly cyclic in the sense of Definition 3.0.3.*

We denote by $\mathcal{C}_{\mathbb{R}} \subset L^2_{\mathbb{R}}(\mathbb{H})$ *the set of weakly* $\mathbb{R}$*-cyclic functions.*

Given a group $\mathbb{K}$ with an even action let $V = \{e, k_0\} \lhd \mathbb{K}$ and define $\mathbb{K}^+ \subset \mathbb{K}$ any set containing e and such that $\#([h] \cap \mathbb{K}^+) = 1$ for any $[h] \in \mathbb{K}/V$. Define the map $B : L^2(\mathbb{K}/V) \to \mathcal{X}$ as

$$Bw(h) = \begin{cases} \overline{w([h])} & \text{if } h \notin \mathbb{K}^+, \\ w([h]) & \text{otherwise.} \end{cases}$$

Obviously B is invertible and $\mathbb{R}$-linear, and thus it endows $L^2(\mathbb{K}/V)$ of the structure of a real vector space. In the case of $\mathbb{Z}_N$, with N even, this amounts to identify $L^2(\mathbb{K}/V)$ with the first $N/2$ components of vectors in $L^2(\mathbb{K}) \cong \mathbb{C}^N$.

Since $\mathcal{X}$ is invariant under the shifts, $S^\ell_{\mathbb{R}} = B^{-1} \circ S^\ell \circ B$ is a representation of $\mathbb{K}$ acting on $L^2(\mathbb{K}/V)$. Taking as a representative of $[h] \in \mathbb{K}/V$ the element $[h] \cap \mathbb{K}^+$, we can describe its action explicitly:

$$S^\ell_{\mathbb{R}} v([h]) = \begin{cases} \overline{v([h-\ell])} & \text{if } ([h] \cap \mathbb{K}^+) - \ell \notin \mathbb{K}^+ \\ v([h-\ell]) & \text{otherwise.} \end{cases}$$

It is then immediate to see that $v \in \mathcal{X}$ is $\mathbb{R}$-cyclic if and only if $\operatorname{span}\{S^k_{\mathbb{R}} B^{-1} v\}_{k \in \mathbb{K}/V} = L^2(\mathbb{K}/V)$. We can then translate the $\mathbb{R}$-cyclicity property to vectors of $L^2(\mathbb{K}/V)$. In particular, when $\mathbb{K}$ is cyclic with N elements, $\mathbb{R}$-cyclicity of $w \in L^2(\mathbb{K}/V) \cong \mathbb{C}^{N/2}$ is equivalent to the invertibility of the following "even circulant" matrix

$$\operatorname{Circ}_{\mathbb{R}} w = \left(w, S_{\mathbb{R}} w, \ldots, S_{\mathbb{R}}^{N/2-1} w \right).$$

4. Wavelet transform

Assume that T is a strongly continuous unitary representation of a locally compact group $\mathbb{G}$ on the Hilbert space $\mathcal{H}_T$. For a vector $\Psi \in \mathcal{H}$, the wavelet transform of $\varphi \in \mathcal{H}$ w.r.t. the wavelet Ψ is

$$W_\Psi \varphi(a) = \langle \varphi, T(a)\Psi \rangle_{\mathcal{H}_T} \qquad \forall a \in \mathbb{G}.$$

Then, W_Ψ is a bounded operator from $\mathcal{H}_T$ to $C_b(\mathbb{G})$.

We call Ψ *admissible* if W_Ψ is an isometry from $\mathcal{H}_T$ into $L^2(\mathbb{G})$ and *weakly admissible* if W_Ψ is a bounded one-to-one mapping into $L^2(\mathbb{G})$.

If $\mathbb{G} = \mathbb{H} \rtimes \mathbb{K}$ is a semi-direct product, it is natural to consider the wavelet transform w.r.t. the quasi-regular representation π, acting on $L^2(\mathbb{H})$. Straightforward computations then show that $W_\Psi \varphi(x, k) = [f \star (R_k \bar{\Psi}(-\cdot))](x)$. We will need the following observation,

$$\mathcal{F}(W_\Psi \varphi(\cdot, k))(\lambda) = \bar{\hat{\Psi}}(R_{-k}\lambda)\, \hat{f}(\lambda), \qquad \forall \lambda \in \widehat{\mathbb{H}}. \tag{5}$$

It is easy (see [3]) to show that

$$\|W_\Psi \varphi\|^2_{L^2(\mathbb{G})} = \int_{\widehat{\mathbb{H}}} |\hat{\varphi}(\lambda)|^2 \|\omega_\Psi(\lambda)\|_{L^2(\mathbb{K})}\, d\lambda. \tag{6}$$

Here, $\omega_\Psi(\lambda)$ is the vector defined at (4). Observe that for any connected $\mathbb{H}$ the Haar measure is σ-finite, and hence the same is true for the Haar measure on $\widehat{\mathbb{H}}$. This implies that $L^1(\widehat{\mathbb{H}})^* = L^\infty(\widehat{\mathbb{H}})$. Since $|\hat{\varphi}|^2 \in L^1(\widehat{\mathbb{H}})$, (6) immediately implies that W_Ψ is a mapping onto $L^2(\mathbb{G})$ if and only if $\lambda \mapsto \|\omega_\Psi(\lambda)\|_{L^2(\mathbb{K})}$ is in $L^\infty(\widehat{\mathbb{H}})$

We then have the following [3].

Theorem 4.0.4. *Let $\Psi \in L^2(\mathbb{H})$. Then,*

- *Ψ is weakly admissible $\iff$ $\lambda \mapsto \|\omega_\Psi(\cdot)\|_{L^2(\mathbb{K})}$ is strictly positive and in $L^\infty(\widehat{\mathbb{H}})$;*
- *Ψ is admissible $\iff$ it holds $\|\omega_\Psi(\lambda)\|_{L^2(\mathbb{K})} = 1$ for a.e. $\lambda \in \widehat{\mathbb{H}}$.*

5. MAP groups and AP functions

For the results in this section we refer to [2, Ch. 16].

Definition 5.0.5. *The* Bohr compactification *of a topological group $\mathbb{G}$ is the universal object $(\mathbb{G}^\flat, \sigma)$ in the category of diagrams $\sigma' : \mathbb{G} \mapsto \mathbb{K}$ where σ' is a continuous homomorphism from $\mathbb{G}$ to a compact group $\mathbb{K}$.*

When $\mathbb{G}$ is abelian, one can construct $\mathbb{G}^\flat$ in the following way: Let $\widehat{\mathbb{G}}_d$ be the Pontryagin dual $\widehat{\mathbb{G}}$ endowed with the discrete topology. Then, its dual is a compact abelian group and it holds $\mathbb{G}^\flat = \widehat{\widehat{\mathbb{G}}_d}$. Moreover, σ is the continuous homomorphism whose dual $\hat{\sigma} : \widehat{\mathbb{G}}_d \to \widehat{\mathbb{G}}$ is the identity map.

As a consequence of the definition, $\sigma(\mathbb{G})$ is dense in $\mathbb{G}^\flat$ and $\mathbb{G}^\flat = \mathbb{G}$ whenever $\mathbb{G}$ is compact. In any case, σ induces a bijection between the finite-dimensional continuous unitary representations of $\mathbb{G}$ and those of $\mathbb{G}^\flat$.

Definition 5.0.6. *If the map $\sigma : \mathbb{G} \to \mathbb{G}^\flat$ is injective, the group $\mathbb{G}$ is said to be* maximally almost periodic *(MAP).*

The group $\mathbb{G}$ is MAP if and only if the continuous finite-dimensional unitary representations of $\mathbb{G}$ separate the points. A connected locally compact group is MAP if and only if it is the direct product of a compact group by $\mathbb{R}^n$. In particular, the Euclidean group of rototranslations $SE(2)$ is not MAP. On the other hand, letting $\mathbb{G}_o$ be the connected component of the identity o, a locally compact group $\mathbb{G}$ such that $\mathbb{G}/\mathbb{G}_o$ is compact is MAP if and only if it is the semidirect product of a compact subgroup $\mathbb{K}$ and of a normal subgroup $\mathbb{H} \cong \mathbb{R}^n$ such that every element of $\mathbb{H}$ commutes with the component of $\mathbb{K}$ containing the identity [2, 16.5.3]. We will only be interested in MAP groups satisfying this property, as is the case for $SE(2, N)$.

Definition 5.0.7. *The set $AP(\mathbb{G})$ of* Bohr almost-periodic functions *over $\mathbb{G}$ is the pull-back through σ of the continuous functions over $\mathbb{G}^\flat$. On the other hand, the set $B_2(\mathbb{G})$ of* Besicovitch almost-periodic functions *over $\mathbb{G}$ is the pull-back through σ of $L^2(\mathbb{G}^\flat)$.*

Equivalently, $f \in AP(\mathbb{G})$ if and only if it is the uniform limit over $\mathbb{G}$ of linear combinations of coefficients of finite-dimensional unitary representations of $\mathbb{G}$. In particular, when $\mathbb{G}$ is abelian, this amounts to say that $f \in AP(\mathbb{G})$ if and only if it is the uniform limit of characters, that is

$$f(x) \sim \sum_{\lambda \in \mathcal{K}_f} a_f(\lambda)\lambda(x),$$

where $\mathcal{K}_f \subset \widehat{\mathbb{G}}$ is a countable set. If $\mathbb{G}$ is MAP then $AP(\mathbb{G})$ is dense in $C(\mathbb{G})$, in the topology of uniform convergence over compact subset.

On the other hand, $f \in B_2(\mathbb{G})$ if and only if it can be written as a square integral linear combination of coefficients of finite-dimensional unitary representations of $\mathbb{G}$. In particular, if the finite-dimensional unitary representations of $\mathbb{G}$ are uncountable, $B_2(\mathbb{G})$ is a non-separable space. In the abelian case, this amounts to say that $f \in B_2(\mathbb{G})$ if and only if

$$f(x) = \sum_{\lambda \in \mathcal{K}_f} a_f(\lambda)\lambda(x) \quad \text{s.t.} \sum_{\lambda \in \mathcal{K}_f} |a_f|^2 < +\infty. \tag{7}$$

We now show the connection between $AP(\mathbb{G})$ and $B_2(\mathbb{G})$ in a more explicit way. Consider the set $C_b(\mathbb{G})$ of continuous bounded functions over $\mathbb{G}$ endowed with the supremum norm. It can be shown that $f \in C_b(\mathbb{G})$ is in $AP(\mathbb{G})$ if and only if $\{\Lambda(a)f\}_{a\in\mathbb{G}}$ is a relatively compact subset of $C_b(\mathbb{G})$. In this case, the convex hull K of $\{\Lambda(a)f\}_{a\in\mathbb{G}}$ in $C_b(\mathbb{G})$ contains exactly one constant function, whose value is called the mean value of f and is denoted by $M(f)$. In the case $\mathbb{G} = \mathbb{R}$, it holds

$$M(f) = \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} f(x)\, dx. \tag{8}$$

Let $f \in AP(\mathbb{G})$ and denote by f' the function of $C(\mathbb{G}^\flat)$ such that $f = f' \circ \sigma$. Then, it holds that $M(f) = \int_{\mathbb{G}^\flat} f'(s)\, ds$, where the integration is taken w.r.t. the Haar measure of total mass equal to 1 on $\mathbb{G}^\flat$. Endowing $AP(\mathbb{G})$ with the sesquilinear form $(f|g) := M(f\bar{g}) = \langle f', g'\rangle$ we obtain a pre-Hilbert space which is canonically isomorphic to $C(\mathbb{G}^\flat)$ regarded as a subspace of $L^2(\mathbb{G}^\flat)$. Since in compact spaces continuous functions are dense in L^2, the closure of $AP(\mathbb{G})$ w.r.t. the induced norm is then $B_2(\mathbb{G})$.

The above shows, in particular, that the pull-back $\sigma^* : L^2(\mathbb{G}^\flat) \to B_2(\mathbb{G})$ is indeed an isomorphism of Hilbert spaces. Thus, in the abelian case, characterization (7) is an immediate consequence of $L^2(\widehat{\mathbb{G}^\flat}) = L^2(\widehat{\mathbb{G}}_d)$.

Remark 5.0.8. *Observe that many non-zero functions on $\mathbb{G}$ are the pull-back of zero functions in $\mathbb{G}^\flat$. In particular it can be proved, and it is a trivial consequence of (8) in the case $\mathbb{G} = \mathbb{R}$, that for any $f \in C_c(\mathbb{G})$ any function $f' : \mathbb{G}^\flat \to \mathbb{C}$ such that $f = f' \circ \sigma$ has to be zero a.e. on $\mathbb{G}^\flat$. Due to this fact, functions in $B_2(\mathbb{G})$ represents indeed equivalence classes of functions $\mathbb{G} \to \mathbb{C}$.*

Let $f \in B_2(\mathbb{G})$ be expressed as in (7), then the following Parseval equality holds

$$(f|f) = \sum_{\lambda \in \mathcal{K}_f} |a_f(\lambda)|^2.$$

As a consequence, the usual diagonalization of the convolution takes place w.r.t. the scalar product $(\cdot|\cdot)$:

$$f \star_{AP} g(x) = \sum_{\lambda \in \mathcal{K}_f \cap \mathcal{K}_g} a_f(\lambda) a_g(\lambda) e^{2\pi i \langle \lambda, x \rangle}.$$

To conclude the section, let us consider the case of $\mathbb{G} = \mathbb{H} \rtimes \mathbb{K}$ under the hypotheses introduced at the beginning of the paper. Then, $\mathbb{G}$ is a MAP group and $\mathbb{G}^\flat = \mathbb{H}^\flat \rtimes \mathbb{K}$, where the action of $\mathbb{K}$ on $\mathbb{H}^\flat$ is obtained through the injection $\sigma_\mathbb{H} : \mathbb{H} \to \mathbb{H}^\flat$ (see [1]). Observe that functions $f \in B_2(\mathbb{G})$ are exactly those such that $f(\cdot, k) \in B_2(\mathbb{H})$ for any $k \in \mathbb{K}$. We will denote by $\sigma_\mathbb{G}$ the injection of $\mathbb{G}$ in $\mathbb{G}^\flat$. With abuse of notation, since $\sigma_\mathbb{G}(x, k) = (\sigma_\mathbb{H}(x), k)$, we will omit the subscript when no confusion arises.

Since whenever $\mathbb{H}$ is non-compact we have that $B_2(\mathbb{H}) \cap L^2(\mathbb{H}) = \{0\}$, the quasi-regular representation is of no use to distinguish the action of $\mathbb{G}$ on $B_2(\mathbb{H})$. Indeed, to define the action of $\mathbb{G}$ of $B_2(\mathbb{H})$ we need to work on the Bohr compactified, as follows. Let $\pi^\flat$ be the quasi-regular representation of $\mathbb{G}$ in $L^2(\mathbb{H}^\flat)$. Since the pull-back $\sigma^* : L^2(\mathbb{H}^\flat) \to B_2(\mathbb{H})$ is an isomorphism, we can consider the representation $\pi^\flat_\sharp$ of $\mathbb{H}^\flat$ on $B_2(\mathbb{H})$. Finally, we pose the following.

Definition 5.0.9. *The* $B_2(\mathbb{H})$-quasi regular representation *of* $\mathbb{G}$ *is* $\pi_{B_2} = \pi^\flat_\sharp \circ \sigma_\mathbb{G}$.

Since it can be shown that $\pi_{B_2}(x, k) f(y) = f(R_{-k}(y - x))$ for any $f \in B_2(\mathbb{H})$ and $(y, h) \in \mathbb{G}$, the $B_2(\mathbb{H})$-quasi regular representation is the correct way to consider the action of $\mathbb{G}$ on Besicovitch almost-periodic functions.

5.1. Subspaces of almost periodic functions

Let $\mathbb{G} = \mathbb{H} \rtimes \mathbb{K}$ satisfy the assumptions of Section 2. Let $E \subset \widehat{\mathbb{H}}$ be invariant under the induced action of $\mathbb{K}$ on $\widehat{\mathbb{H}}$ and consider the set

$$\mathcal{T}(E) \coloneqq \operatorname{span}\{\lambda \in E\} \subset B_2(\mathbb{H}).$$

That is, by (7), a function $f \in \mathcal{T}(E)$ has the form

$$f(x) = \sum_{\lambda \in \mathcal{K}_f} a_f(\lambda)\, \lambda(x), \qquad \text{where } \mathcal{K}_f \subset E \text{ and } \sum_{\lambda \in \mathcal{K}_f} |a_f(\lambda)|^2 < +\infty. \tag{9}$$

Observe that, if E is countable $\mathcal{T}(E)$ is a separable subspace of $B_2(\mathbb{H})$ while if E is finite $\mathcal{T}(E)$ is a finite dimensional subspace of $B_2(\mathbb{H})$

Clearly for any $f \in \mathcal{T}(E)$ it holds that $\omega_f(\lambda) = (a_f(R_{-k}\lambda))_{k \in \mathbb{K}} = 0$ for any $\lambda \notin E$. Thus, whenever $|\widehat{\mathbb{H}} \setminus E| \neq 0$, no $f \in \mathcal{T}(E)$ is weakly cyclic or weakly $\mathbb{R}$-cyclic.

Definition 5.1.1. *Let $E \subset \widehat{\mathbb{H}}$ be invariant under the induced action of $\mathbb{K}$. A function $f \in \mathcal{T}(E)$ is* AP-weakly cyclic *if $\omega_f(\lambda)$ is cyclic for all $\lambda \in E$. Similarly, a real valued function $f \in \mathcal{T}(E)$ is* AP-weakly $\mathbb{R}$-cyclic *if $\omega_f(\lambda)$ is $\mathbb{R}$-cyclic for all $\lambda \in E$.*

The sets of AP-weakly cyclic and AP-weakly $\mathbb{R}$-cyclic functions are denoted respectively by $\mathcal{C}^{AP}$ and $\mathcal{C}_{\mathbb{R}}^{AP}$.

In order for the bispectral invariants defined in Part III to make sense on $\mathcal{T}(E)$ we need to make some assumptions on the set E. Namely, let $I^{\otimes} \subset E \times E$ be the set of couples (λ_1, λ_2) such that $\lambda_1 + R_k\lambda_2 \in E$ for any $k \in \mathbb{K}$, and pose the following.

Definition 5.1.2. *The set $E \subset \widehat{\mathbb{H}}$ is* bispectrally admissible *if it is invariant under the induced action of $\mathbb{K}$ and if $E = F \cup G$ with $F \times F \subset I^{\otimes}$ and such that for any $\lambda \in G$ it holds $\lambda = \lambda_1 + R_k\lambda_2$ for some $\lambda_1, \lambda_2 \in F$ and $k \in \mathbb{K}$.*

6. Functional spaces under consideration

In this section we introduce the two functional spaces we are interested in: compactly supported real-valued square-integrable functions on the plane, which model images, and Besicovitch almost-periodic functions on the plane, which model textures.

We will then discuss the correct mathematical framework for image recognition.

6.1. Compactly supported square-integrable functions on the plane

Let $D_R \subset \mathbb{R}^2$ be the compact disk of radius $R > 0$. For fixed $R > 0$, the size of the screen, images are elements of

$$\mathcal{V}(D_R) = \{f \in L^2_{\mathbb{R}}(\mathbb{R}^2) \mid \exists c \in \mathbb{R}^2 \text{ s.t. } \operatorname{supp} f \subset c + D_R \text{ and } \operatorname{avg} f \neq 0\}.$$

Recall that the set $\mathcal{C}_{\mathbb{R}}$ is the set of weakly $\mathbb{R}$-cyclic $L^2_{\mathbb{R}}(\mathbb{R}^2)$ functions, defined in Section 3. The following can be proved by using the same argument of the third case in Theorem 10.0.7.

Theorem 6.1.1. *For any $R > 0$ the set $\mathcal{C}_{\mathbb{R}} \cap \mathcal{V}(D_R)$ is open dense in $\mathcal{V}(D_R)$.*

We now define a centering operator for images, in the sense of Definition 2.0.4. We let

$$\mathcal{I} = \left\{ f \in L^2_{\mathbb{R}}(\mathbb{R}^2) \cap L^1_{\mathbb{R}}(\mathbb{R}) \mid \operatorname{avg} f \neq 0 \right\}.$$

This is a closed subspace of $L^2_{\mathbb{R}}(\mathbb{R}^2) \cap L^1_{\mathbb{R}}(\mathbb{R})$ with open and dense complement. Then, for $f \in \mathcal{I}$, the *geometric center of f* is the point $\operatorname{cent}(f) = (x_1^f, x_2^f)$ defined as

$$x_j^f = \frac{1}{\operatorname{avg}(f)} \int_{\mathbb{R}^2} x_j\, f(x)\, dx.$$

The centering operator $\Phi_c : \mathcal{I} \to \mathcal{I}$ is defined as $\Phi_c(f) = \tau_{\operatorname{cent}(f)} f$, so that $\Phi_c(f)$ has always center in the origin.

Since $\mathcal{V}(D_R) \subset \mathcal{I}$, using this centering we obtain the following identification

$$\mathcal{V}(D_R) \cong (\mathbb{R}^2 \oplus \{f \in L^2_{\mathbb{R}}(D_R) \mid \operatorname{cent}(f) = 0\})/\sim,$$

where $(c_1, f) \sim (c_2, g)$ if and only if either $f = g = 0$ or $f = g$ and $c_1 = c_2$. That is, a couple $(c, f) \in \mathcal{V}(\mathbb{R}^2)$ is composed of the actual image f and its center c.

6.2. AP functions on the plane

The space $B_2(\mathbb{G})$ of Besicovitch almost-periodic functions on a topological group $\mathbb{G}$ has been introduced in Section 5. We will consider $B_2(\mathbb{R}^2)$ as a model of textures. Recall that to distinguish the action of $SE(2,N)$ of $B_2(\mathbb{R}^2)$ we have to use the $B_2(\mathbb{R}^2)$-quasi regular representation, introduced in Definition 5.0.9

6.2.1. Subsets of almost-periodic functions on the plane

When considering textures, due to the finiteness of the screen, we will need to restrict ourselves to certain subsets of $B_2(\mathbb{R}^2)$. This is achieved by considering the space $\mathcal{T}(E)$, where E is a bispectrally admissible set, introduced in Section 5.1.

Let us denote $\mathcal{T}_{\mathbb{R}}(E) \subset \mathcal{T}(E)$ the set of real valued functions in $\mathcal{T}(E)$. Observe that $\mathcal{T}_{\mathbb{R}}(E) \neq \varnothing$ only if $E = -E$.

Theorem 6.2.1. *When E is finite, the set $\mathcal{C}^{AP}$ is open and dense in $\mathcal{T}(E)$. Moreover, when E is countable the set $\mathcal{C}^{AP}$ is residual. The same results are true for the set $\mathcal{C}^{AP}_{\mathbb{R}}$ w.r.t. $\mathcal{T}_{\mathbb{R}}(E)$, when $E = -E$.*

Proof. We start by claiming that the set $\mathcal{V}$ of cyclic vectors in $\mathbb{C}^N$ is open and dense. The openness follows from the fact that circulant matrices are diagonalized by the discrete Fourier transform (unitary) matrix $\mathcal{F}_N$. Indeed, this yields

$$\{v \in \mathbb{C}^N \mid v \text{ is cyclic}\} = \mathcal{F}_N^* \left(\{\hat v \in \mathbb{C}^N \mid \hat v_j \neq 0 \,\forall j\}\right) = \mathcal{F}_N^* \left(\bigcap_{j=0}^{N-1} \{\hat v \in \mathbb{C}^N \mid \hat v_j \neq 0\}\right),$$

which proves that $\{v \in \mathbb{C}^N \mid v \text{ is cyclic}\}$ is open since it is the inverse image under an isometry of a finite intersection of open sets. The density follows by observing that $\operatorname{Circ}(v+w) = \operatorname{Circ} v + \operatorname{Circ} w$ and that, if A, B are two matrices with A invertible and B not invertible, since $\varepsilon \mapsto \det(\varepsilon A + B)$ is analytic, then $\varepsilon A + B$ is invertible for all $\varepsilon > 0$ sufficiently small. This completes the proof of the claim.

Since E is invariant under the action of $\mathbb{Z}_N$, we can write $E \cong \tilde E \times \mathbb{C}^N$, where $\tilde E \subset E$ contains only one representative per orbit. Then,

$$\mathcal{C}^{\mathrm{AP}} = \bigcap_{\lambda \in \tilde E} \{f \mid \omega_f(\lambda) \text{ is cyclic.}\}.$$

Since by the previous claim the sets on the r.h.s. are open and dense, and since $\tilde E$ is countable or finite depending on whether E is, this completes the proof of the statement regarding $\mathcal{C}^{\mathrm{AP}}$.

The proof of the statement regarding $\mathcal{C}^{\mathrm{AP}}_{\mathbb{R}}$ follows from similar arguments. □

The usual procedure we employ to construct a bispectrally admissible set is the following:

1. Fix a finite rotationally invariant set $E_1 \subset \widehat{\mathbb{R}^2}$.
2. Recursively define $E_j = \{\lambda + \mu \mid \lambda, \mu \in E_{j-1}\} \cup E_{j-1}$.
3. Stop after $M \in \mathbb{N} \cup \{+\infty\}$ steps and let $E = E_M$.

An important case is when $M = +\infty$ and the starting set E_1 is chosen to be the set of the N-th roots of unity, i.e., $E_1 = \{e^{2\pi i k/N} \mid k = 0, \dots, N-1\}$. When $N = 2$ this yields $E = \mathbb{Z} \subset \mathbb{R}^2$, while for $N = 3, 4, 6$ the set E turns out to be one of the possible lattices of $\mathbb{R}^2$. Moreover, we have the following.

Proposition 6.2.2. *If N is even then the set E obtained from the above procedure, with $M = +\infty$ and $E_1 = \{e^{2\pi i k/N} \mid k = 0, \dots, N-1\}$, is a countable additive subgroup of $\mathbb{R}^2$ on which $\mathbb{Z}_N$ acts.*

Proof. The fact that E is countable is a consequence of the construction. Let us prove that it is a subgroup of $\mathbb{R}^2$. By construction, for any $\lambda, \mu \in E$ we have that $\lambda + \mu \in E$, so the set is closed w.r.t. addition. Moreover, $0 \in E$, since $0 = 1 - 1$ and $1, -1 \in E_1$ by parity of N. Finally, let us prove by induction that for any $x \in E_n$ it holds that $-x \in E_n$, which will complete the proof. Clearly, again by parity of N, this is true for $n = 1$. Assume this to be true for n, and observe that $\nu \in E_{n+1}$ if and only if $\nu = \lambda + \mu$ for $\lambda, \mu \in E_n$. Then, $-\nu = -\lambda - \mu \in E_{n+1}$ as well, completing the proof of the claim.

A simple induction procedure shows that E is rotationally invariant, and hence that the action of $\mathbb{Z}_N$ restricts to it, which completes the proof. □

The following proposition clarifies what happens for not necessarily even values of N.

Proposition 6.2.3. *Let $N \geq 5$ but $N \neq 6$. Then, the set E obtained from the above procedure, starting with $E_1 = \{e^{2\pi i k/N} \mid k = 0, \dots, N-1\}$ and with $M = +\infty$, is dense in $\mathbb{R}^2$.*

Proof. For $\theta \in \mathbb{S}^1$, let us denote with $L_\theta \subset \mathbb{R}^2$ the line passing through the origin forming an angle θ with the x-axis. Obviously, $L_0 = \mathbb{R} \times \{0\}$.

In the following we will use these two well-known facts:

- For any irrational number α, the set $\alpha\mathbb{Z} + \mathbb{Z}$ is dense in $\mathbb{R}$.
- For any $N \geq 5$ and $N \neq 6$, $\cos(2\pi/N)$ is irrational.

We divide the proof in three steps.

1. *Let $\theta_1, \theta_2 \in \mathbb{S}^1$ and $A \subset L_{\theta_1}$, $B \subset L_{\theta_2}$ be two dense subset. Then, for $\theta = (\theta_2 - \theta_1)/2$ the set $(A + B) \cap L_\theta$ is dense in L_θ:* Without loss of generality we can assume $\theta_1 = 0$, and thus $\theta_2 = 2\theta$. Let us define $f : L_0 \to L_\theta$ by $f(p) = p + R_{2\theta} p$, which is clearly bicontinuous. Thus, $f(A)$ is dense in L_θ and to complete the proof it

suffices to show that $\overline{f(A)} \subset \overline{A+B}$. To this aim, let $y = f(a) \in \overline{f(A)}$ and consider a sequence $(a_n)_n \subset A$ such that $a_n \to a$. Then, $f(a_n) = a_n + R_{2\theta} a_n \to f(a)$, where $R_{2\theta} a_n \in L_{2\theta}$. For any a_n let us consider $(b_{n,k})_k \subset B$ such that $b_{n,k} \to R_{2\theta} a_n$ as $k \to +\infty$. Then, we have

$$\lim_n a_n + b_{n,n} = \lim_n a_n + R_{2\theta} a_n = f(a) = y.$$

This proves that $\overline{f(A)} \subset \overline{A+B}$, completing the proof.

2. *If* $\cos(2\pi/N)$ *is irrational, then* $E \cap L_0$ *contains a dense subgroup of* $\mathbb{R}$*:* Clearly, by construction of E, $\mathbb{Z} \subset E \cap L_0$. Simple trigonometric considerations yield $e^{2\pi i/N} + e^{-2\pi i/N} = (2\cos(2\pi/N), 0) \in E_2 \subset E$. This implies that $2\cos(2\pi/N)\mathbb{Z} \subset E \cap L_0$. Hence, $2\cos(2\pi/N)\mathbb{Z} + \mathbb{Z} \subset E \cap L_0$. Using then the above cited well-known facts, we complete the proof of the claim.

3. *The set* E *is dense in* $\mathbb{R}^2$*:* Consider the set $V \subset \mathbb{S}^1$ obtained with the following iterative procedure. Fix $V_1 = \{2\pi k/N \mid k = 0, \dots, N-1\}$ and then $V_n = \{(\theta - \theta')/2 \mid \theta, \theta' \in V_{n-1}\} \cup V_{n-1}$. Finally, $V = \bigcup_{n\in\mathbb{N}} V_n$. It is easy to prove that V is a dense subset of $\mathbb{S}_1$. Moreover, by the previous steps of the proof, we have that for any $\theta \in V$ it holds that $S \cap L_\theta$ is dense in L_θ. Indeed, $E \cap L_0$ is dense in L_0 by the previous step and, for any k, $E \cap L_{2\pi k/N}$ is dense in $L_{2\pi k/N}$ by rotational invariance of E which yields the claim using the first step of the proof.

   Let us show that E is dense proceeding by contradiction. Assume that there exists an open set $U \subset \mathbb{R}^2$ such that $U \cap E = \varnothing$. Consider the set $P = \{\theta \in \mathbb{S}^1 \mid L_\theta \cap U \neq \varnothing\}$. It is easy to see that P is open in $\mathbb{S}^1$, which implies that $V \cap P \neq \varnothing$. Thus, for some $\theta \in V$ we have $L_\theta \cap U \neq \varnothing$. Since $L_\theta \cap U$ is open in L_θ and $E \cap L_\theta$ is dense, we finally have that $E \cap U \cap L_\theta \neq \varnothing$ which contradicts the assumption $E \cap U = \varnothing$, completing the proof. □

6.2.2. Centering almost periodic functions

In the final part of the paper, in order to be able to restrict only to the action of $\mathbb{Z}_N$, we will need to quotient out the effect of translations on $B_2(\mathbb{R}^2)$. Here, given a *finite* rotationally invariant set $E \subset \widehat{\mathbb{R}^2}$ and a compact subset K of $\mathbb{R}^2$, we define a centering of an appropriate subset of $\mathcal{T}(E) \subset B_2(\mathbb{R}^2)$ w.r.t. K, in the sense of Definition 2.0.4. This centering is obtained exploiting the fact that functions of $\mathcal{T}(E)$ are restrictions of periodic functions on a bigger space.

Let us denote by M the number of orbits under the action of $\mathbb{Z}_N$ contained in E and fix any orbit-based enumeration $\{\lambda_k^j\}_{k,j}$ of E. That is, the index $j \in \{0, \dots, M-1\}$ parametrizes the orbits and $\lambda_{k+h}^j = R_h \lambda_k^j$ for any $k, h \in \mathbb{Z}_N$. Consider the following subspace of the space of functions on the torus $\mathbb{T}^{MN}$:

$$\widetilde{\mathcal{T}}(E) = \operatorname{span}\left\{ z \mapsto e^{2\pi i z_k^j} \mid k \in \{0, \dots, N-1\} \text{ and } j \in \{0, \dots, M-1\} \right\}.$$

Since both $\mathcal{T}(E)$ and $\widetilde{\mathcal{T}}(E)$ are NM-dimensional Hilbert spaces, they are isomorphic. Let us explicitly describe the isomorphism $\Gamma : \mathcal{T}(E) \to \widetilde{\mathcal{T}}(E)$. By (9) we know that any $f \in \mathcal{T}(E)$ can be written as

$$f(x) = \sum_{j=0}^{M-1} \sum_{k\in\mathbb{Z}_N} f_k^j e^{2\pi i \langle \lambda_k^j, x\rangle}, \quad \forall x \in \mathbb{R}^2.$$

Then, we have

$$\Gamma f(z) = \sum_{j=0}^{M-1} \sum_{k\in\mathbb{Z}_N} f_k^j e^{2\pi i z_k^j}, \quad \forall z = (z_k^j) \in \mathbb{T}^{MN}.$$

Moreover, letting $\Xi : x \in \mathbb{R}^2 \mapsto (\langle \lambda_k^j, x\rangle)_{k,j} \in \mathbb{T}^{MN}$, we have $\Gamma^{-1}\tilde f = \tilde f|_{\Xi(\mathbb{R}^2)} \circ \Xi$ for any $\tilde f \in \widetilde{\mathcal{T}}(E)$.

The following proposition is a direct consequence of the definition of Γ.

Proposition 6.2.4. *For any $h \in \mathbb{K}$, the image under Γ of the rotation by h operator, $\tilde R_h = \Gamma \circ R_h \circ \Gamma^{-1} \in \mathcal{U}(\widetilde{\mathcal{T}}(E))$, is given by*

$$\tilde R_h \tilde f(z) = \sum_{j=0}^{M-1} \sum_{k\in\mathbb{Z}_N} \tilde f_{k-h} e^{2\pi i z_k^j}, \quad \forall \tilde f \in \widetilde{\mathcal{T}}(E).$$

Moreover, for any $y \in \mathbb{R}^2$, the image under Γ of the translation by y operator, $\tilde\tau_y = \Gamma \circ \tau_y \circ \Gamma^{-1} \in \mathcal{U}(\mathcal{T}(E))$, is given by

$$\tilde\tau_y \tilde f(z) = \tilde f(z - \Xi(y)), \quad \forall \tilde f \in \widetilde{\mathcal{T}}(E).$$

Let us consider the following subset of $\mathcal{T}(E)$:

$$\mathcal{I} = \left\{ f \in \mathcal{T}(E) \mid \text{the maximum of } \Re(\Gamma f) \text{ is attained at exactly one point } m(f) \in \mathbb{T}^{MN} \right\}.$$

In Lemma 6.2.9 we prove that $\mathcal{I} = \{f \in \mathcal{T}(E) \mid f_k^j \neq 0, \quad \forall k, j\}$, and hence is open dense in $\mathcal{T}(E)$. For $f \in \mathcal{I}$, by the previous proposition it is clear that $m(\tau_y f) = m(f) - \Xi(y)$.

Proposition 6.2.5. *The set $X := \{\Xi(y) \mid y \in \mathbb{R}^2\}$ is a 2-dimensional submanifold of $\mathbb{T}^{MN}$. Moreover, whenever $N \neq 2, 3, 4, 6$, its closure is a 4-dimensional torus.*

Proof. □

The above proposition shows that, in general, $0 \notin m(f) - X$. Even worst, due to the density property, the function $\xi \in X \mapsto \|m(f) - \xi\|$ could not even attain a minimum. For this reason, we need to restrict our attention to a bounded set of translations.

Fix a compact $K \subset \mathbb{R}^2$. Since $\{\Xi(y) \mid y \in K\}$ is a closed subset of $\mathbb{T}^{MN}$ the value $\min_{y\in K} \|m(f) - \Xi(y)\|$ is attained in at least one point. We then let

$$\mathcal{R}_K := \left\{ f \in \mathcal{I} \mid \min_{y\in K} \|m(f) - \Xi(y)\| \text{ is realized only at one point of } \mathbb{R}^2 \right\} \subset \mathcal{T}(E).$$

Definition 6.2.6. *The* center *of* $f \in \mathcal{R}_K$ *is the point* $\operatorname{cent}(f) \in \mathbb{R}^2$ *such that* $\|m(f) - \operatorname{cent}(f)\| = \min_{y\in\mathbb{R}^2} \|m(f) - \Xi(y)\|$. *The* centering *of* $\mathcal{R}_K$ *w.r.t.* K *is then the function* $\Phi : \mathcal{R}_K \to \mathcal{R}_K$ *defined by* $\Phi(f) = \tau_{\operatorname{cent}(f)} f$.

Remark 6.2.7. *Here we centered w.r.t. the maximum of the real part of the function* Γf *since we cannot define the geometric center of a function in* $\widetilde{\mathcal{T}}(E)$. *Indeed, it is easy to see that all functions in* $\widetilde{\mathcal{T}}(E)$ *have zero average.*

We conclude this section by proving that the set where the centering is defined is indeed quite big.

Proposition 6.2.8. *The set* $\mathcal{R}_K$ *is residual in* $\mathcal{T}(E)$.

Proof. □

Lemma 6.2.9. *For* $\tilde f \in \widetilde{\mathcal{T}}(E)$, $\Re\tilde f$ *assumes its maximum value in exactly one point of* $\mathbb{T}^{MN}$ *if and only if* $\tilde f_k^j \neq 0$ *for any* $k \in \mathbb{Z}_N$ *and* $j \in \{0, \dots, M-1\}$.

Moreover, letting $\tilde f_k^j = \rho_k^j e^{2\pi\theta_k^j}$ *we have that* $m(f) = (-\theta_k^j)_{k,j}$.

Proof. From the definition of $\widetilde{\mathcal{T}}(E)$ and the addition formula of the cosine, follows that

$$\begin{aligned}
\Re\tilde f(z) &= \sum_{j=0}^{M-1} \sum_{k\in\mathbb{Z}_N} \left(\Re(\tilde f_k^j)\cos\left(2\pi z_k^j\right) - \Im(\tilde f_k^j)\sin\left(2\pi z_k^j\right)\right) \\
&= \sum_{j=0}^{M-1} \sum_{k\in\mathbb{Z}_N} \underbrace{\rho_k^j \cos\left(2\pi(\theta_k^j + z_k^j)\right)}_{=:\varphi_k^j(z_k^j)}.
\end{aligned}$$

Since all the φ_k^j depends of different variables, it is clear that the maximum is realized at those points $\bar z = (\bar z_k^j)$ such that

$$\varphi_k^j(\bar z_k^j) = \max_{\xi\in\mathbb{T}} \varphi_k^j(\xi), \quad \forall j \in \{0, \dots, M-1\},\ k \in \mathbb{Z}_N.$$

Clearly, if $\tilde f_k^j = 0$ for some k, j then $\varphi_k^j \equiv 0$, proving the only if part of the statement. On the other hand, if $\tilde f_k^j \neq 0$ then φ_k^j has exactly one maximum in $\bar z_k^j = -\theta_k^j$, completing the proof of the lemma. □

6.2.3. Polar almost-periodic expansion

Here, given a finite set $F \subset \mathbb{R}^2$, we present a procedure to represent a function $f : F \to \mathbb{C}$ as an element of $\mathcal{T}(E)$, where E is a finite bispectrally admissible set.

Let us fix some notation. We denote elements of F and E in polar coordinates, as

$$E = \left\{\left(\lambda_k, \omega_k + \frac{2\pi r}{N}\right) \in \mathbb{R}_+ \times [0, 2\pi) \mid r = 0, \dots, N-1,\ k \in \{0, \dots, |E|/N - 1\}\right\},$$
$$F = \left\{\left(\rho_j, \alpha_j + \frac{2\pi r}{N}\right) \in \mathbb{R}_+ \times [0, 2\pi) \mid r = 0, \dots, N-1,\ j \in \{0, \dots, |F|/N - 1\}\right\}.$$

We also denote $f_{j,\theta}(r) := f\left(\rho_j, \alpha_j + \frac{2\pi r}{N} + \theta\right)$, still in polar coordinates, and $a_f(k,r)$ the Fourier component of $f \in \mathcal{T}(E)$ corresponding to $\left(\lambda_k, \omega_k + \frac{2\pi r}{N}\right) \in E$, as per (9). Finally, we let $\widehat{f_{j,\theta}}$ to be the Fourier transform of the vector $f_{j,\theta} \in \mathbb{C}^N$ and $\widehat{a_f(k)}$ that of $(a_f(k,r))_{r=0}^{N-1} \in \mathbb{C}^N$. That is,

$$\widehat{f_{j,\theta}}(n) = \sum_{r=0}^{N-1} f_{j,\theta}\, e^{-i\frac{2\pi}{N}nr} \quad \text{and} \quad \widehat{a_f(k)}(n) = \sum_{r=0}^{N-1} a_f(k,r)\, e^{-i\frac{2\pi}{N}nr}. \tag{10}$$

We start with the following.

Definition 6.2.10. *The* discrete Bessel functions of the first kind *are the functions*

$$J_n^\alpha(\lambda) := \sum_{k=0}^{N-1} e^{i\left(\lambda\cos\left(\alpha+\frac{2\pi k}{N}\right)-n\frac{2\pi k}{N}\right)}, \qquad n \in \mathbb{N},\ \alpha \in \left[0, \frac{2\pi}{N}\right), \lambda \in \mathbb{R}.$$

Observe that, as $N \to +\infty$, the discrete Bessel functions converge uniformly to $i^n J_n(\lambda) = J_{-n}(\lambda)$, where J_{-n} is the Bessel function of order $-n$. This can be easily checked using the integral representation

$$J_n(\lambda) = \int_0^1 e^{i(\lambda\sin\theta - n\theta)}\, d\theta.$$

Proposition 6.2.11. *Let $E \subset \widehat{\mathbb{R}^2}$ and $F \subset \mathbb{R}^2$ be two finite rotationally invariant set. Then, for any $f \in \mathcal{T}(E)$, any $\theta \in [0, \frac{2\pi}{N})$, any $n \in \{0,\ldots,N-1\}$, and any $j \in \{0,\ldots,|F|/N-1\}$ it holds*

$$\widehat{f_{j,\theta}}(n) = \left(\mathcal{J}_\theta^n\, \widehat{a_f(\cdot)}(n)\right)_j, \qquad \text{where } \mathcal{J}_\theta^n := \left(J_n^{\alpha_k-\omega_k-\theta}(\lambda_k\rho_j)\right)_{j,k}. \tag{11}$$

Here, we denoted with $\widehat{a_f(\cdot)}(n)$ the vector $k \mapsto \widehat{a_f(k)}(n)$.

Proof. In the following computations to lighten the notation we will omit the extrema of the sums. The sums in k will always be between 0 and $|E|/N-1$, while those in r and h will be between 0 and $N-1$.

Since $f \in \mathcal{T}(E)$, for any $x \in \mathbb{R}^2$ whose polar coordinates are $(\rho_j, \alpha_j + \frac{2\pi}{N}h + \theta)$ it holds

$$f(x) = f_{j,\theta}(h) = \sum_{k,r} a_f(k,r) e^{i\rho_x\lambda_k\cos\left(\alpha_x+\theta-\omega_k-\frac{2\pi r}{N}\right)}.$$

By (10) we have

$$\begin{aligned}
\widehat{f_{j,\theta}}(n) &= \sum_{k,r,h} a_f(k,r) e^{i\left(\rho_x\lambda_k\cos\left(\alpha_x+\theta-\omega_k+\frac{2\pi}{N}(h-r)\right)-\frac{2\pi}{N}nh\right)} \\
&= \sum_{k,r} a_f(k,r) e^{-in\frac{2\pi r}{N}} \sum_h e^{i\left(\rho_x\lambda_k\cos\left(\alpha_x+\theta-\omega_k-\frac{2\pi}{N}(h-r)\right)-n\frac{2\pi}{N}(h-r)\right)} \\
&= \sum_{k,r} a_f(k,r) e^{-in\frac{2\pi}{N}r} J_n^{\alpha_j-\omega_k-\theta}(\lambda_k\rho_j) \\
&= \sum_{k,r} \widehat{a_f(k)}(n)\, J_n^{\alpha_j-\omega_k-\theta}(\lambda_k\rho_j),
\end{aligned}$$

which proves the statement. □

It is very important to observe that the matrices $\mathcal{J}_\theta^n$ defined in Proposition 6.2.11 depend *only* on the two sets E, F. We have thus justified the following.

Definition 6.2.12. *Let $E \subset \widehat{\mathbb{R}^2}$ and $F \subset \mathbb{R}^2$ be two finite rotationally invariant sets such that the matrices $\mathcal{J}^n := \mathcal{J}_0^n$ are invertible for any $n \in \{0, \dots, N-1\}$. Given a discrete function $f : F \to \mathbb{C}$ the* polar almost-periodic expansion *of f is the function $\tilde{f} \in \mathcal{T}(E)$ whose Fourier components $a_{\tilde{f}}(k, r)$ satisfy* (11).

Observe that from Proposition 6.2.11 immediately follows that $f = \tilde{f}$ on F.

Remark 6.2.13. *Let $M := |E|/N$. For $\mathcal{J}^n$ to be invertible it is clearly necessary that $|F| = |E|$. Then, since the inverses of the matrices $\mathcal{J}^n$ can be precomputed, the Fourier components of the polar almost-periodic expansion of $f : F \to \mathbb{C}$ can be computed via $2M$ discrete Fourier transforms (FFT's) and N products of $M \times M$ matrices.*

Part II. Lifts

Let $\mathbb{G} = \mathbb{H} \rtimes \mathbb{K}$ be a semi-direct product, as considered in Section 2. Here, we are interested in operators $L : L^2(\mathbb{H}) \to L^2(\mathbb{G})$, which we call *lift operators* for obvious reasons. Observe that, via the isomorphism $\sigma^* : L^2(\mathbb{H}) \to B_2(\mathbb{H}^\flat)$ any lift $L : L^2(\mathbb{H}^\flat) \to L^2(\mathbb{G}^\flat)$ induces a lift $L' : B_2(\mathbb{H}) \to B_2(\mathbb{G})$ of Besicovitch almost periodic functions.

We are mainly interested in identifying the action of the quasi-regular representation on $f \in L^2(\mathbb{H})$ by analyzing the Fourier transform of the lift $Lf \in L^2(\mathbb{G})$. Thus, we focus on lifts interwining the quasi-regular representation acting on $L^2(\mathbb{H})$ with the left regular representations on $L^2(\mathbb{G})$.

7. Left-invariant lifts

The natural class of lifts to consider is the following.

Definition 7.0.14. *A lift operator $L : L^2(\mathbb{H}) \to L^2(\mathbb{G})$ is* left-invariant *if $\Lambda(a) \circ L = L \circ \pi(a)$ for any $a \in \mathbb{G}$.*

It is clear from the definition that for an injective left-invariant lift it holds

$$Lf = \Lambda(a)Lg \iff f = \pi(a)g.$$

In the sequel we will thus be mainly interested in injective left-invariant lifts.

Obviously, any wavelet transform via the quasi-regular representation induces a left-invariant lift operator. As presented in Section 4 the injectivity of these lift is equivalent to the existence of a weakly admissible vector. Later on we will call these *regular left-invariant lift operators.*

It is readily seen from Theorem 4.0.4, that if $\widehat{\mathbb{H}}$ is not compact, no admissible vector exists for the quasi-regular representation of $\mathbb{G}$. Thus, in this case no regular left-invariant lift can be an isometry.

Remark 7.0.15. *Closed invariant subspaces of π are characterized in (2), and in particular they are of the form $\mathcal{A} = \mathcal{A}_U$ where $U \subset \widehat{\mathbb{H}}$ is a $\mathbb{K}$ invariant measurable set. From Theorem 4.0.4 it follows the existence of admissible vectors for any sub-representation π_U of π with $|U| < +\infty$. By considering restrictions of regular left-invariant lift to these subsets, it is then possible to obtain isometric lifts. However, due to the Paley-Wiener Theorem, none of these $\mathcal{A}_U$ contains the compactly supported functions when $\widehat{\mathbb{H}}$ is non-compact.*

We now characterize those left-invariant lifts that come from wavelet transforms, showing that most "reasonable" injective left-invariant lifts are of this type.

Theorem 7.0.16. *Let $L : L^2(\mathbb{H}) \to C(\mathbb{G}) \cap L^2(\mathbb{G})$ be a* linear *left-invariant lift such that $f \mapsto Lf(o,e)$ is a continuous function from $\mathcal{A}$ to $\mathbb{C}$. Then, there exists $\Psi \in L^2(\mathbb{H})$ such that $\lambda \mapsto \|\omega_\Psi(\lambda)\|_{L^2(\mathbb{K})}$ is essentially bounded on $\widehat{\mathbb{H}}$ and*

$$Lf(a) = W_\Psi f(a) \, (= \langle f, \pi(a)\Psi\rangle) \qquad \forall a \in \mathbb{G}. \tag{12}$$

Moreover, L is injective if and only if $\lambda \mapsto \|\omega_\Psi(\lambda)\|_{L^2(\mathbb{K})}$ is strictly positive.

Proof. By the assumptions, $f \mapsto Lf(o,e)$ is an element of the dual $L^2(\mathbb{H})^*$, which by the Riesz theorem can be identified with $L^2(\mathbb{H})$. Thus, there exists $\Psi \in L^2(\mathbb{H})$ such that $Lf(o,e) = \langle f, \Psi\rangle$. Formula (12) is then obtained from the left-invariance of L. Indeed, by this and the unitariety of π, for any $a \in \mathbb{G}$ it holds

$$Lf(a) = \Lambda(a^{-1})Lf(o,e) = L(\pi(a^{-1})f)(o,e) = \langle \pi(a^{-1})f, \Psi\rangle = \langle f, \pi(a)\Psi\rangle.$$

Finally, the fact that $\lambda \mapsto \|\omega_\Psi(\lambda)\|_{L^2(\mathbb{K})}$ is in $L^\infty(\widehat{\mathbb{H}})$ is a consequence of the discussion following (6), while the last statement is a consequence of Theorem 4.0.4. □

Remark 7.0.17. *In the above theorem, we could have assumed the function $f \mapsto Lf(o,e)$ to be continuous from $C_c(\mathbb{H})$ (or $C_0(\mathbb{H})$) to $\mathbb{C}$. Due to the characterization of the dual of $C_c(\mathbb{H})$ (or $C_0(\mathbb{H})$), this would have yield the same result with the wavelet Ψ being a finite (or locally finite) Radon measure on $\mathbb{H}$.*

The trivial lift considered in [8] is obtained in a similar way, choosing $\Psi = \delta_o$, the Dirac delta mass centered at the identity of $\mathbb{H}$. Observe that this choice does not guarantee range $L \subset C(\mathbb{G})$.

Definition 7.0.18. *A left-invariant lift L is* regular *if it satisfies the assumptions of Theorem 7.0.16 and is injective.*

7.1. Gabor wavelets

7.2. Fourier transforms of lifted functions

Proposition 7.2.1. *Let $L : L^2(\mathbb{H}) \to L^2(\mathbb{G})$ be a regular left-invariant lift. Then, it holds*

$$\widehat{Lf}(T^\lambda) = \overline{\omega_f(\lambda)}^* \otimes \overline{\omega_\Psi(\lambda)} \in \mathrm{HS}(L^2(\mathbb{K})) \qquad \forall f \in L^2(\mathbb{H}),\, \lambda \in \widehat{\mathbb{H}}.$$

Moreover, if $\Psi \in L^1(\mathbb{H})$ it holds

$$\widehat{Lf}(T^{\hat{o}\times\hat{k}}) = \begin{cases} \operatorname{avg}(f)\, \operatorname{avg}(\bar\Psi) & \text{if } \hat{k} = \hat{e}, \\ 0 & \text{otherwise.} \end{cases} \qquad \forall f \in L^2(\mathbb{H}) \cap L^1(\mathbb{H}),\, \hat{k} \in \widehat{\mathbb{K}}.$$

Proof. By Proposition 2.0.7, Theorem 7.0.16 and (5), we have

$$\begin{aligned} \widehat{Lf}(T^\lambda).v(h) &= \int_{\mathbb{K}} \mathcal{F}[Lf(\cdot, h-k)](R_{-k}\lambda)\, v(k)\, dk \\ &= \int_{\mathbb{K}} \mathcal{F}[\langle f, \pi(\cdot, h-k)\Psi\rangle](R_{-k}\lambda)\, v(k)\, dk \\ &= \hat{\bar\Psi}(R_{-h}\lambda) \int_{\mathbb{K}} \hat{f}(R_{-k}\lambda) v(k)\, dk. \end{aligned}$$

This completes the proof of the first part of the statement.

On the other hand, observe that $a_{Lf} \equiv \operatorname{avg} f \operatorname{avg} \bar\Psi$. Indeed,

$$\int_{\mathbb{H}} \pi(x, \ell)\Psi(y)\, dx = \operatorname{avg}\Psi \qquad \forall \ell \in \mathbb{K},\, y \in \mathbb{H}.$$

Thus, the second part of the statement follows from Proposition 2.0.7, observing that $\widehat{a_{Lf}}(\hat{e}) = \operatorname{avg} f \operatorname{avg} \bar\Psi$ and $\widehat{a_{Lf}}(\hat{k}) = 0$ for $\hat{k} \neq \hat{e}$. □

Remark 7.2.2. *The above result implies that the Fourier transform of a a lifted function has rank at most 1. In particular, it is far from being invertible.*

Corollary 7.2.3. *Let $L : L^2(\mathbb{H}) \to L^2(\mathbb{G})$ be a regular left-invariant lift. Then, for any $\lambda_1, \lambda_2 \in \widehat{\mathbb{H}} \setminus \{\hat{o}\}$ and any $f \in L^2(\mathbb{H})$, we have*

$$A \circ \widehat{Lf}(T^{\lambda_1} \otimes T^{\lambda_2}) \circ A^* = \bigoplus_{k\in\mathbb{K}} \left(\overline{\omega_f(\lambda_1 + R_k\lambda_2)}^* \otimes \overline{\omega_\Psi(\lambda_1 + R_k\lambda_2)} \right).$$

Here, A is the equivalence from $L^2(\mathbb{K}\times\mathbb{K})$ to $\bigoplus_{k\in\mathbb{K}} L^2(\mathbb{K})$ defined in (3).

Proof. The statement follows from the Induction-Reduction Theorem, the fact that the Fourier transform commutes with equivalences and direct sums, and Proposition 7.2.1. □

8. Cyclic lift

In [8] to overcome the difficulties presented by non-invertible Fourier transforms, a different lift operator (called cyclic lift) is considered. In this section we put those ideas in a more general context.

A close analysis of the cyclic lift of [8], shows that it is the composition of two operators, that we will discuss in the following sections.

8.1. Almost left-invariant lifts

The first problem to overcome when building a lift that can yield invertible Fourier transforms, is to avoid left-invariance.

Definition 8.1.1. *A lift operator $L : L^2(\mathbb{H}) \to L^2(\mathbb{G})$ is* almost left-invariant *if $\Lambda(y,h)Lf(x,k) = L(\pi(R_k y, 2h)f)(x,k)$ for all $(x,k),(y,h) \in \mathbb{G}$.*

From the definition, it immediately follows that injective almost left-invariant lifts satisfy

$$\begin{aligned} Lf = \Lambda(o,h)Lg \iff f = \pi(o,2h)g \\ Lf(\cdot,e) = \Lambda(y,e)Lg(\cdot,e) \iff f = \pi(y,e)g. \end{aligned} \tag{13}$$

Observe that the second equivalence holds only for $Lf(\cdot,e)$. The fact that it cannot be extended to $k \neq e$ implies that the invariants of almost left-invariant lifts cannot separate the action of translations on $L^2(\mathbb{H})$. To overcome this problem we will later introduce cyclic lifts.

The following theorem (similar to Theorem 7.0.16) justifies the above seemingly random definition.

Theorem 8.1.2. *Let $L : L^2(\mathbb{H}) \to C(\mathbb{G}) \cap L^2(\mathbb{G})$ be a linear almost left-invariant lift such that $f \mapsto Lf(o,e)$ is a continuous function from $L^2(\mathbb{H})$ to $\mathbb{C}$. Then, there exists $\Psi \in L^2(\mathbb{H})$ satisfying*

$$\lambda \mapsto \int_{\mathbb{K}} |\hat{\Psi}(R_{2k}\lambda)|^2 \, dk \text{ is essentially bounded on } \widehat{\mathbb{H}}, \tag{14}$$

and such that

$$Lf(x,k) = \langle R_{-k}f, \pi(x,k)\Psi \rangle \qquad \forall (x,k) \in \mathbb{G}. \tag{15}$$

Moreover, L is injective if and only if the function in (14) is strictly positive.

Proof. As in Theorem 7.0.16, from the Riesz representation theorem follows immediately that $Lf(o,e) = \langle f, \Psi \rangle$ for some $\Psi \in L^2(\mathbb{H})$. Then, (15) follows by writing $Lf(x,k) = \Lambda(x,k)^{-1}Lf(o,e)$ and using the definition of almost-left invariance.

To prove (14), observe that by (15) we have $Lf(x,k) = [(R_{-2k}f) \star (-\bar{\Psi})](R_{-k}x)$. Here, we denoted with $\star$ the group convolution on $\mathbb{H}$. Then, it follows that

$$\mathcal{F}(Lf(\cdot,k))(\lambda) = \bar{\hat{\Psi}}(R_{-k}\lambda)\, \hat{f}(R_k\lambda), \qquad \forall \lambda \in \widehat{\mathbb{H}}. \tag{16}$$

This allows to compute, via the Parseval identity,

$$\|Lf\|_{L^2(\mathbb{G})} = \int_{\mathbb{K}} \int_{\widehat{\mathbb{H}}} |\hat{f}(R_k\lambda)|^2 |\hat{\Psi}(R_{-k}\lambda)|^2 \, d\lambda \, dk = \int_{\widehat{\mathbb{H}}} |\hat{f}(\mu)|^2 \int_{\mathbb{K}} |\hat{\Psi}(R_{-2k}\mu)|^2 \, d\mu \, dk$$

Since $(L^1(\widehat{\mathbb{H}}))^* = L^\infty(\widehat{\mathbb{H}})$, this implies (14). Moreover, it also implies that $\ker L = \{0\}$ if and only if the function in (14) is positive. By linearity of L this implies the last statement. □

Remark 8.1.3. *As in the case of Theorem 7.0.16, in the above we could have assumed the function $f \mapsto Lf(o,e)$ to be continuous from $C_c(\mathbb{H})$ (or $C_0(\mathbb{H})$) to $\mathbb{C}$. This would have yielded similar results with Ψ being a finite (or locally finite) Radon measure on $\mathbb{H}$.*

Definition 8.1.4. *An almost left-invariant lift is* regular *if it satisfies the assumptions of Theorem 8.1.2 and is injective.*

Let us remark that the conditions on Ψ for an almost left-invariant lift to be regular coincide with those for left-invariant lifts (obtained in Theorem 7.0.16) if and only if the map $k \mapsto k + k$ is a bijection on $\mathbb{K}$. If $\mathbb{K} = \mathbb{Z}_N$, this is equivalent to N being odd.

8.2. Fourier transform of lifted functions

Proposition 8.2.1. *Let $L : L^2(\mathbb{H}) \to L^2(\mathbb{G})$ be a regular almost left-invariant lift and let $f \in L^2(\mathbb{H})$. Then, for any $\lambda \in \widehat{\mathbb{H}} \setminus \{\hat{o}\}$, it holds that $\widehat{Lf}(T^\lambda) \in HS(L^2(\mathbb{K}))$ has matrix elements*

$$\widehat{Lf}(T^\lambda)_{i,j} = \bar{\hat{\Psi}}(R_{-i}\lambda)\hat{f}(R_{i-2j}\lambda) = \overline{\omega_\Psi(\lambda)_i}\, \omega_f(\lambda)_{2j-i}.$$

Moreover, if $\Psi \in L^1(\mathbb{H})$ it holds

$$\widehat{Lf}(T^{\hat{o}\times\hat{k}}) = \begin{cases} \operatorname{avg}(f)\, \operatorname{avg}(\bar{\Psi}) & \text{if } \hat{k} = \hat{e}, \\ 0 & \text{otherwise,} \end{cases} \qquad \forall f \in L^2(\mathbb{H}) \cap L^1(\mathbb{H}),\ \hat{k} \in \widehat{\mathbb{K}}.$$

Proof. The second part of the statement can be proved exactly as in Proposition 7.2.1.

By Proposition 2.0.7 and (16), for any $v \in L^2(\mathbb{K})$ we have

$$\widehat{Lf}(T^\lambda).v(h) = \int_{\mathbb{K}} \mathcal{F}[Lf(\cdot, h-k)](R_{-k}\lambda)\, v(k)\, dk = \bar{\hat{\Psi}}(R_{-h}\lambda) \int_{\mathbb{K}} \hat{f}(R_{h-2k}\lambda) v(k)\, dk.$$

This proves the first part of the statement, completing the proof of the proposition. □

The following will be useful in Section 12.

Proposition 8.2.2. *Let $f \in L^2(\mathbb{H})$. Then, for any $\lambda_1, \lambda_2 \in \widehat{\mathbb{H}}$ it holds that*

$$\begin{aligned} A \circ \widehat{Lf}(T^{\lambda_1} \otimes T^{\lambda_2}) \circ A^* &= \bigoplus_{k\in\mathbb{K}} \left(\bar{\hat{\Psi}}(R_{-i}(\lambda_1 + R_k\lambda_2))\hat{f}(R_{i-2j}(\lambda_1 + R_k\lambda_2)) \right)_{i,j\in\mathbb{K}} \\ &= \bigoplus_{k\in\mathbb{K}} \left(\overline{\omega_\Psi(\lambda_1 + R_k\lambda_2)_i}\, \omega_f(\lambda_1 + R_k\lambda_2)_{2j-i} \right)_{i,j\in\mathbb{K}}. \end{aligned}$$

Here, A is the equivalence from $L^2(\mathbb{K}\times\mathbb{K})$ to $\bigoplus_{k\in\mathbb{K}} L^2(\mathbb{K})$ defined in (3).

Proof. The statement follows from the Induction-Reduction Theorem, the fact that the Fourier transform commutes with equivalences and direct sums, and Proposition 8.2.1. □

Proposition 8.2.3. *Let L be a regular almost left-invariant lift such that the associated wavelet satisfies $\hat{\Psi} \neq 0$ a.e. on $\widehat{\mathbb{H}}$. Then, the following dichotomy holds*

1. *The action of $\mathbb{K}$ on $\mathbb{H}$ is even in the sense of Definition 3.1.1 and $\widehat{Lf}(T^\lambda)$ is at most of rank $N/2$;*

2. *The action of $\mathbb{K}$ on $\mathbb{H}$ is not even and for a.e. $\lambda \in \widehat{\mathbb{H}} \setminus \{\hat{o}\}$, the invertibility of $\widehat{Lf}(T^\lambda)$ is equivalent to that of the matrix $\operatorname{Circ} \omega_f(\lambda)$.*

Proof. Fix $\lambda \in \widehat{\mathbb{H}} \setminus \{\hat{o}\}$ and let $E, F \in \mathbb{C}^{N\times N}$ be the matrices with elements $E_{i,j} = \bar{\hat{\Psi}}(R_{-i}\lambda)$ and $F_{i,j} = \hat{f}(R_{i-2j}\lambda)$. Observe in particular that the column $F_{\cdot,j}$ is $i \mapsto \omega_f(R_{-2j}\lambda)_{-i}$. Theorem 8.2.1 implies that $\widehat{Lf}(T^\lambda) = E \odot F$, where we denoted with $\odot$ the element-wise product between matrices. Due to the special form of E, this implies that

$$\det \widehat{Lf}(T^\lambda) = \left(\Pi_{i\in\mathbb{K}} \bar{\hat{\Psi}}(R_{-i}\lambda)\right) \det F.$$

By the assumption on $\hat{\Psi}$, this implies that, outside a zero measure set, the matrix $\widehat{Lf}(T^\lambda)$ is invertible if and only F is invertible.

If the action of $\mathbb{K}$ on $\mathbb{H}$ is even, then using an argument similar to the proof of Proposition 3.1.2 it is easy to see that $F_{\cdot,j} = \overline{F_{\cdot,j+k_0}}$. This proves the first part of the dichotomy. On the other hand, it is clear that when the action is not even, there exists a permutation $\sigma : (0, \dots, N-1) \mapsto (2j \mod N)_{j=0}^{N-1}$. Then, letting $\nu : k \mapsto -k$, we have $\nu \circ F \circ \sigma = \operatorname{Circ} \omega_f(\lambda)$ and the statement is proved. □

8.3. Cyclic lifts

Cyclic lifts are obtained by composing almost left-invariant lifts with the centering operators defined in Definition 2.0.4, in order to quotient out the action of $\mathbb{H}$ from $L^2(\mathbb{H})$.

Definition 8.3.1. *Let $\mathcal{A} \subset L^2(\mathbb{H})$ be invariant under the action of π and $U \subset \mathbb{H}$. A lift operator $L : \mathcal{A} \to L^2(\mathbb{G})$ is a* cyclic lift *if there exist a centering Φ of $\mathcal{A}$ w.r.t. U and an almost left-invariant lift P such that $L = P \circ \Phi$.*

From the definition of centering and from (13), it immediately follows that, whenever P is injective,

$$Lf = \Lambda(0,k)Lg \iff f = \pi(x,k)g \text{ for some } x \in U \subset \mathbb{H}. \tag{17}$$

Thus, if $U = \mathbb{H}$ and $\mathbb{K}$ is even in the sense of Section 3.1, a cyclic lift can be used to separate translations and rotations. Together with Proposition 8.2.3 this is the second reason why, when $\mathbb{K} = \mathbb{Z}_N$, we will need to assume its cardinality to be odd.

Definition 8.3.2. *A cyclic lift $L = P \circ \Phi$ is* regular *if P is a regular almost-invariant lift.*

The following is immediate, from Theorem 8.1.2.

Corollary 8.3.3. *Let $L = P \circ \Phi : \mathcal{A} \to C(\mathbb{G}) \cap L^2(\mathbb{G})$ be a regular cyclic lift. Then, there exists $\Psi \in L^2(\mathbb{H})$ satisfying*

$$\lambda \mapsto \int_{\mathbb{K}} |\hat{\Psi}(R_{2k}\lambda)|^2 \, d\lambda \text{ is strictly positive and essentially bounded on } \widehat{\mathbb{H}}, \tag{18}$$

and such that

$$Lf(x,k) = \langle R_{-k}\Phi(f), \pi(x,k)\Psi\rangle \qquad \forall (x,k) \in \mathbb{G}.$$

Moreover, L is injective if and only if the function in (18) is positive.

Remark 8.3.4. *The cyclic lift considered in [8] is obtained by choosing Φ to be the centering discussed in Section 6.1 defined for $\mathcal{A} = \mathcal{V}(D_R)$ and letting $\Psi = \delta_o$, the Dirac delta mass centered at the identity of $\mathbb{H}$ (see Remark 8.1.3).*

Part III. Bispectrum

9. Definition of bispectral invariants

Let $\mathbb{G}$ be an unimodular group. The maps $f \mapsto I_f$ are called *invariants* for $\mathbb{G}$ if $I_f = I_{\Lambda(a)f}$ for any $a \in \mathbb{G}$. A choice of invariants is *complete* if it separates the orbits of Λ. That is, if for any $f, g \in L^2(\mathbb{G})$ we have

$$I_f = I_g \iff f = \Lambda(a)g \text{ for some } a \in \mathbb{G}.$$

A choice of invariants is *weakly complete* if the above is true only on some residual subset of $L^2(\mathbb{G})$.

The first invariants that one can consider are the following.

Definition 9.0.5. *The* (power) spectrum invariants *of $f \in L^2(\mathbb{G})$ are the set $I^1_f = \{I^1_f(T) \mid T \in \operatorname{supp} \mu_{\widehat{\mathbb{G}}}\}$, where*

$$I^1_f(T) = \widehat{f}(T) \circ \widehat{f}(T)^*.$$

The power spectrum invariants are not weakly complete even in the simple case of $\mathbb{G} = \mathbb{R}$. In this case $I_f(\lambda) = |\hat{f}(\lambda)|^2$ for any $\lambda \in \operatorname{supp} \mu_{\widehat{\mathbb{G}}} = \widehat{\mathbb{R}}$, and it is easy to build a counterexample. Indeed, it suffices to fix some $\phi : \widehat{\mathbb{R}} \to \mathbb{S}^1$ and consider the function $g = \mathcal{F}^{-1}(e^{i\phi(\lambda)}\hat{f}(\lambda))$. Clearly, g is such that $I^1_f = I^1_g$ but $f = \Lambda(a)g$ if and only if $\phi(\lambda) = a\lambda$.

Thus, we need to consider richer sets of invariants, as the following.

Definition 9.0.6. *The* (power) bispectral invariants *of* $f \in L^2(\mathbb{G})$ *are the set* $I_f^2 = \{I_f^2(T_1, T_2) \mid T_1, T_2 \in \operatorname{supp} \mu_{\widehat{\mathbb{G}}}\}$*, where*

$$I_f^2(T_1, T_2) = \widehat{f}(T_1) \otimes \hat{f}(T_2) \circ \hat{f}(T_1 \otimes T_2)^*. \tag{19}$$

A priori one needs to use both the bispectral invariants and the power spectrum invariants, although we will see that in most cases, and in particular in the case $\mathbb{G} = SE(2, N)$, we have that $I_f^2 \supset I_f^1$.

10. General results

In this section we will prove the weak completeness of the bispectral invariants in three different cases. In all these situations, we indeed prove that the bispectral invariants are complete on the following (residual) subset of $L^2(\mathbb{G})$:

$$\mathcal{G} := \left\{ f \in L^2(\mathbb{G}) \mid \begin{array}{l} f \text{ has compact support and } \hat{f}(T) \text{ is invertible on} \\ \text{an open and dense subset of } \operatorname{supp} \hat{\mu}_{\mathbb{G}} \end{array} \right\} \tag{20}$$

The following result guarantees that in the cases under consideration $\mathcal{G}$ is a sufficiently large set.

Theorem 10.0.7. *The following hold:*

1. *If* $\mathbb{G}$ *is a connected abelian Lie group,* $\mathcal{G}$ *is the set of all non-zero compactly supported functions of* $L^2(\mathbb{G})$*.*
2. *If* $\mathbb{G}$ *is a compact and separable,* $\mathcal{G}$ *is residual.*
3. *If* $\mathbb{G} = \mathbb{H} \rtimes \mathbb{K}$*, under the assumptions of Section 2 with* $\mathbb{H}$ *connected Lie group,* $\mathcal{G}$ *is residual. If, moreover,* $\mathbb{H} = \mathbb{R}^N$ *then* $\mathcal{G}$ *is open and dense in the set of compactly supported functions of* $L^2(\mathbb{G})$*.*

Proof. Case 1: If $\mathbb{G}$ is a connected abelian Lie group, it holds that $\mathbb{G} \cong \mathbb{R}^N \times \mathbb{T}^M$, and so its dual is isomorphic to $\widehat{\mathbb{G}} \cong \mathbb{R}^N \times \mathbb{Z}^M$. By the Paley-Wiener Theorem, for any $f \in L^2(\mathbb{G})$ with compact support the function $\hat{f}(\cdot, k)$ is analytic for any $k \in \mathbb{Z}^M$. Thus, $f \in \mathcal{G}$ if and only if for any $k \in \mathbb{Z}^M$ it holds that $\hat{f}(\lambda, k) \neq 0$ for on an open and dense subset $\lambda \in \widehat{\mathbb{R}^N}$, property which is satisfied by every non-zero analytic function. This proves that $\mathcal{G}$ contains all non-zero compactly supported functions of $L^2(\mathbb{G})$.

Case 2: If $\mathbb{G}$ is compact separable, then $\widehat{\mathbb{G}}$ is countable and discrete (see, e.g., [2]) and thus

$$\mathcal{G} = \left\{ f \in L^2(\mathbb{G}) \mid \hat{f}(\lambda) \text{ is invertible for all } \lambda \in \widehat{\mathbb{G}} \right\}.$$

Then, for any fixed λ the set of those $f \in L^2(\mathbb{G})$ such that $\hat{f}(T)$ is invertible is open and dense. Thus, $\mathcal{G}$ is the countable intersection of open and dense sets and hence residual.

Case 3: Let $f \in L^2(\mathbb{G})$ and denote $f_k = f(\cdot, k)$ for any $k \in \mathbb{K}$. Since $\mathbb{H} \cong \mathbb{R}^N \times \mathbb{T}^M$ and $\widehat{\mathbb{H}} \cong \widehat{\mathbb{R}^N} \times \mathbb{Z}^M$, if f is compactly supported the functions $\lambda \in \widehat{\mathbb{R}^N} \mapsto f_k(\lambda, h)$ are analytic for any $h \in \mathbb{Z}^M$.

We now prove that, if $\mathbb{H} \cong \mathbb{R}^N$, the set $\mathcal{G}$ is open and dense in the set of compactly supported functions. By Proposition 2.0.7 the k-th diagonal of $\hat{f}(T^\lambda)$ is $\omega_{f_k}(\lambda)$ and hence $\lambda \mapsto \det \hat{f}(T^\lambda)$ is analytic. In particular, $f \in \mathcal{G}$ if and only if there exists λ_0 such that $\det \hat{f}(T^{\lambda_0}) \neq 0$.

Observe that $\mathcal{G} \neq \varnothing$. Indeed, it suffices to fix $\lambda_0 \in \widehat{\mathbb{R}^N}$ and consider f such that $f_k \equiv 0$ for any $k \neq e$ and f_e is such that no component of $\omega_{f_e}(\lambda)$ is zero. This ensures that $\hat{f}(T^{\lambda_0})$ is invertible and hence that $f \in \mathcal{G}$.

To prove that $\mathcal{G}$ is dense, let us fix $f \in \mathcal{G}$ and consider $g \notin \mathcal{G}$. Then, for some λ_0 such that $\hat{f}(T^{\lambda_0})$ is invertible it holds that $\widehat{g+\varepsilon f}(T^{\lambda_0})$ is invertible for any $\varepsilon > 0$ sufficiently small[2]. This proves that $g + \varepsilon f \in \mathcal{G}$ and thus that $g \in \overline{\mathcal{G}}$.

Let us now prove that $\mathcal{G}$ is open. Fix $f \in \mathcal{G}$ and consider a sequence of compactly supported functions $f_n \to f$ in $L^2(\mathbb{G})$. This implies that $\hat{f}_n \to \hat{f}$ in $L^2(\mathbb{G})$ and thus in measure. In particular, $\hat{f}_n(T^\lambda) \to \hat{f}(T^\lambda)$ in measure and hence for any n sufficiently big there exists λ_0 such that $\hat{f}_n(T^{\lambda_0}) \neq 0$. This implies that $f_n \in \mathcal{G}$ for any n sufficiently big, and hence that $\mathcal{G}$ is open.

The result for the case $\mathbb{H} \cong \mathbb{R}^N \times \mathbb{T}^M$ follows by considering the sets $\mathcal{G}_h$, $h \in \mathbb{Z}^M$, of compactly supported functions whose Fourier transforms $\hat{f}(T^{(\lambda,h)})$ are invertible for an open and dense set of $\lambda \in \widehat{\mathbb{R}^N}$. The same arguments as above can be used to prove that $\mathcal{G}_h$ is open and dense. Finally, since $\mathcal{G} = \bigcap_{h\in\mathbb{Z}^M} \mathcal{G}_h$, this proves that $\mathcal{G}$ is residual. □

10.1. Abelian group

Let $\mathbb{G}$ be an abelian group. Then all its representations are one dimensional and the Plancherel measure is the Haar measure on the character group $\widehat{\mathbb{G}}$. In this case, the set $\mathcal{G}$ defined in (20), becomes

$$\mathcal{G} = \left\{ f \in L^2(\mathbb{G}) \mid f \text{ has compact support and } \hat{f}(\lambda) \neq 0 \text{ for all } \lambda \in \widehat{\mathbb{G}} \right\}$$

Simple computations shows that

$$I^1_f(\lambda) = |\hat{f}(\lambda)|^2 \qquad \text{and} \qquad I^2_f(\lambda_1, \lambda_2) = \hat{f}(\lambda_1)\hat{f}(\lambda_2)\bar{\hat{f}}(\lambda_1 + \lambda_2). \tag{21}$$

In this case, we have that $I^2_f \supset I^1_f$ for any $f, g \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$. Indeed, observe that choosing $\lambda_1 = \lambda_2 = \hat{o}$ in the bispectral invariants yields $\operatorname{avg}(f)|\operatorname{avg}(f)|^2 = \operatorname{avg}(g)|\operatorname{avg}(g)|^2$, which implies that $\operatorname{avg}(f) = \operatorname{avg}(g)$. This shows that $I^2_f(\lambda_1, 0) = I^1_f(\lambda_1)$.

Theorem 10.1.1. *The bispectral invariants are complete on the set $\mathcal{G}$. In particular, if $\mathbb{G}$ is either compact separable or a connected Lie group, they are weakly complete on compactly supported functions.*

Proof. The second part of the statement is a direct consequence of Theorem 10.0.7. Let then $f, g \in \mathcal{G}$ be such that $I^2_f = I^2_g$. Since this implies that $I^1_f = I^1_g$, we have that

[2] This follows from the linearity of the Fourier transform and the analyticity of the map $\varepsilon \mapsto \det(A+\varepsilon B)$ where A, B are matrices.

$|\hat{f}| = |\hat{g}|$. Thus $\hat{f}$ and $\hat{g}$ vanish on the same set $\mathcal{I}$. Moreover, observe that since f and g are compactly supported, their Fourier transforms $\hat{f}$ and $\hat{g}$ are continuous.

Let $u(\lambda) = \hat{g}(\lambda)/\hat{f}(\lambda)$ for any $\lambda \in \mathcal{I}$. Since u is the ratio of two continuous functions who vanish only on a discrete set, it is measurable. Then, by the equality of the bispectral invariants and (21) it follows that u satisfies

$$u(\lambda_1 + \lambda_2) = u(\lambda_1)u(\lambda_2).$$

This implies that u is a measurable character of $\hat{\mathbb{G}}$ and thus, by a well-known result [5, Theorem 22.17], has to be continuous. By Pontryagin duality this proves the existence of $a \in \mathbb{G}$ such that $u(\lambda) = \lambda(a)$. Thus, we have proved that $\hat{f}(\lambda) = \lambda(a)\hat{g}(\lambda)$, which by Theorem 1.2.2 implies that $f = \Lambda(a)g$, completing the proof. □

In the case $\mathbb{G} = \mathbb{R}^n$ the above result can be strengthened.

Corollary 10.1.2. *The bispectral invariants on $\mathbb{R}^n$ are complete on compactly supported functions of $L^2(\mathbb{R}^2)$.*

Proof. It suffices to observe that by the Paley-Wiener Theorem Fourier transforms of compactly supported functions are analytic. Since analytic non-zero functions have a discrete zero-level set, this implies that the set $\mathcal{G}$ of Theorem 10.1.1 coincide with all the considered functions. □

10.2. Compact group

Let $\mathbb{G}$ be a compact separable group. In this case the set of irreducible unitary representations is endowed with the discrete topology and thus the set $\mathcal{G}$ defined in (20) becomes

$$\mathcal{G} = \left\{ f \in L^2(\mathbb{G}) \mid \hat{f}(T) \text{ is invertible for any } T \in \widehat{\mathbb{G}} \right\}. \tag{22}$$

With the same arguments used for abelian groups, it is possible to show that $I_f^2 \supset I_f^1$. We then have the following.

Theorem 10.2.1. *The bispectral invariants are weakly complete on $\mathbb{G}$. More precisely, they discriminate on the above defined set $\mathcal{G}$.*

Proof. The fact that $\mathcal{G}$ is residual in $L^2(\mathbb{G})$ follows from Theorem 10.0.7 Let $f, g \in \mathcal{G}$ be such that $I_f^2 = I_g^2$. The idea of the proof is to show that this allows to build a quasi-representation u of $\widehat{\mathbb{G}}$ such that $\hat{f}(T) \circ u(T) = \hat{g}(T)$. The conclusion then will follow by Chu (or Tannaka) duality.

- *Step 1 - Definition of the candidate quasi-representation:* Since $I_f^2 = I_g^2$ implies that $I_f^1 = I_g^1$, it holds that $\hat{f}(T) \circ \hat{f}(T)^* = \hat{g}(T) \circ \hat{g}(T)^*$ for all $T \in \widehat{\mathbb{G}}$. By invertibility of $\hat{f}(T)$ we can define $u(T) = \hat{f}(T)^{-1} \circ \hat{g}(T)$ for any $T \in \operatorname{supp} \mu_{\widehat{\mathbb{G}}}$. Moreover, since $I_f^2 = I_g^2$ implies the equality of the bispectral invariants (19) for any (nonnecessarily irreducible) unitary representation, the same definition holds for any representation in $\operatorname{Rep}(\mathbb{G})$, the Chu dual of $\mathbb{G}$.

- *Step 2 - u is indeed a quasi-representation:* Let us start by checking that $u(T)$ is unitary. This follows from the equality of the first invariants. Indeed,

$$u(T)^* u(T) = \hat{g}(T)^* \left(\hat{f}(T)\hat{f}(T)^*\right)^{-1} \hat{g}(T) = \mathrm{Id}\,.$$

  We now check the properties of the quasi-representations.

  1. *Commutation with the direct sum:* This follows from the definition of $u(T)$ and the analogous property of the Fourier transform.
  2. *Commutation with the tensor product:* From the equality of the bispectral invariants and the definition of u, for all $T_1, T_2 \in \widehat{\mathbb{G}}$ we obtain

$$\hat{f}(T_1) \otimes \hat{f}(T_2) \circ \hat{f}(T_1 \otimes T_2)^* = \\ \hat{f}(T_1) \otimes \hat{f}(T_2) \circ u(T_1) \otimes u(T_2) \circ u(T_1 \otimes T_2)^* \circ \hat{f}(T_1 \otimes T_2)^*.$$

     Since $\hat{f}(T)$ is invertible for all $T \in \widehat{\mathbb{G}}$, this and the unitariety of u imply $u(T_1) \otimes u(T_2) = u(T_1 \otimes T_2)$.
  3. *Commutation with the equivalences:* Again, this follows from the definition of $u(T)$ and the analogous property of the Fourier transform.
  4. *Continuity:* The sets $\mathrm{Rep}_n(\mathbb{G})$ are discrete, due to compactness of $\mathbb{G}$, hence this is trivial.

  Thus, u is a quasi-representation of $\widehat{\mathbb{G}}$.

- *Step 3 - Chu duality:* By Theorem 1.3.2, the group $\mathbb{G}$ has the Chu duality property. Thus, being u a quasi-representation, there exists $a \in \mathbb{G}$ such that for all $T \in \widehat{\mathbb{G}}$ it holds $u(T) = T(a)$. Then, $\hat{g}(T) = \hat{f}(T) \circ T(a)$ for all $T \in \operatorname{supp} \mu_{\widehat{\mathbb{G}}} \subset \widehat{\mathbb{G}}$ which, by Theorem 1.2.2 implies that $f = \Lambda(a)g$, completing the proof. □

10.3. Moore groups that are semi-discrete products

We now consider the semi-direct product $\mathbb{G} = \mathbb{H} \rtimes \mathbb{K}$ introduced in Section 2, where $\mathbb{K}$ is finite with N elements. Since $L^2(\mathbb{K}) \cong \mathbb{C}^N$, the group $\mathbb{G}$ is a Moore group and the set $\mathcal{G}$ becomes

$$\mathcal{G} = \left\{ f \in L^2(\mathbb{G}) \mid \hat{f}(T^\lambda) \text{ is invertible for } \lambda \text{ in an open dense set of } \widehat{\mathbb{H}} \right\}. \tag{23}$$

The model to keep in mind for $\mathbb{G}$ is the semi-discrete roto-translation group of the plane $SE(2, N)$, where $\mathbb{H} = \mathbb{R}^2$, $\mathbb{K} = \mathbb{Z}_N$ and R_k is the rotation of angle $2\pi k/N$.

Due to the explicit structure of the group we can compute the expression of the invariants.

Proposition 10.3.1. *Let $f \in L^2(\mathbb{G})$. Then, for all $\lambda, \lambda_1, \lambda_2 \in \widehat{\mathbb{H}} \setminus \{\hat{o}\}$ and any $k, \ell \in \mathbb{K}$,*

$$I^1_f(T^\lambda) = \left(\sum_{h \in \mathbb{K}} \hat{f}(T^\lambda)_{i,h} \, \overline{\hat{f}(T^\lambda)_{j,h}} \right)_{i,j \in \mathbb{K}}$$

$$(A \circ I^2_f(T^{\lambda_1} \otimes T^{\lambda_2}) \circ A^*)_{k,\ell} = \left(\sum_{h \in \mathbb{K}} \hat{f}(T^{\lambda_1})_{i,h} \, \hat{f}(T^{\lambda_2})_{i-\ell,h-k} \, \overline{\hat{f}(T^{\lambda_1 + R_k \lambda_2})_{j,h}} \right)_{i,j \in \mathbb{K}}.$$

Proof. The first part of the statement follows immediately from the definition of the invariants. To prove the second part, it suffices to use the Induction-Reductin Theorem and the properties of the equivalence A proved in Proposition 2.1.2. □

Observe that similarly to the abelian case, it is easy to show that

$$I^2_f \supset \{ I^1_f(T) \mid T = T^\lambda \text{ for } \lambda \neq \hat{o} \text{ or } T = T^{\hat{o} \times \hat{e}} \}.$$

Theorem 10.3.2. *The bispectral invariants are complete on the set $\mathcal{G}$ defined in (23). In particular, if $\mathbb{H}$ is a connected Lie group, they are weakly complete on the set of compactly supported $L^2(\mathbb{G})$ functions.*

Proof. The last part of the statement follows from Theorem 10.0.7. Let us consider $f, g \in \mathcal{G}$ such that $I^2_f = I^2_g$. The idea of the proof is similar to the one of Theorem 10.2.1. Namely, we start by defining a candidate quasi-representation U. Here, however, we will not prove that U is a quasi-representation, since it is possible, and simpler, to directly prove that $U(T) = T(a)$ for some $a \in \mathbb{G}$.

Due to the added complexities arising in this case, we have delayed the technical parts of the proof to later lemmata, contained in Section 10.3.0.1.

- *Step 1 - Definition of the candidate quasi-representation:* From $I^2_f = I^2_g$ it follows that the sets where $\hat{f}$ and $\hat{g}$ fails to be invertible are the same. We will denote it with I. We then let

  $$U(T^\lambda) = \hat{f}(T^\lambda)^{-1} \hat{g}(T^\lambda) \in \mathbb{C}^{N \times N} \qquad \forall \lambda \in I. \tag{24}$$

  Clearly, $U(T^\lambda)$ is unitary for any $\lambda \in I$ (this can be proved as in step 2 of the proof of Theorem 10.2.1).

  Since $\lambda \mapsto \hat{f}(T^\lambda)$ and $\lambda \mapsto \hat{g}(T^\lambda)$ are measurable, and $\hat{G} \setminus I$ is open and dense, by (24) also $\lambda \mapsto U(T^\lambda)$ is measureable on I.

  By the equality of the second-type invariants and the definition of U, for any $\lambda_1, \lambda_2 \in I$ it holds

  $$\hat{f}(\lambda_1) \otimes \hat{f}(\lambda_2) \circ \hat{f}(T^{\lambda_1} \otimes T^{\lambda_2})^* = \hat{f}(\lambda_1) \otimes \hat{f}(\lambda_2) \circ U(T^{\lambda_1}) \otimes U(T^{\lambda_2}) \circ \hat{g}(T^{\lambda_1} \otimes T^{\lambda_2})^*.$$

  By the invertibility of $\hat{f}(\lambda_1) \otimes \hat{f}(\lambda_2)$, this yields

  $$\hat{f}(T^{\lambda_1} \otimes T^{\lambda_2}) \circ U(T^{\lambda_1}) \otimes U(T^{\lambda_2}) = \hat{g}(T^{\lambda_1} \otimes T^{\lambda_2}). \tag{25}$$

- *Step 2 - The function $\lambda \mapsto U(T^\lambda)$ is continuous on I:* This is done in Lemma 10.3.3.
- *Step 3 - The function $\lambda \mapsto U(T^\lambda)$ can be extended to a continuous function on $\widehat{\mathbb{H}} \setminus \{\hat{o}\}$ for which (25) is still true:* This is done in Lemma 10.3.4.
- *Step 4 - There exists $a \in \mathbb{G}$ such that $U(T^\lambda) = T^\lambda(a)$ for any $\lambda \in \widehat{\mathbb{H}} \setminus \{\hat{o}\}$:* This is done in Lemma 10.3.5.
- *Step 5 - It holds that $\Lambda(a)f = g$:* By definition of U and Theorem 2.0.5, the previous step proves that $\hat{f}(T^\lambda) \circ T^\lambda(a) = \hat{g}(T^\lambda)$ for any $\lambda \in \widehat{\mathbb{H}} \setminus \{\hat{o}\}$. By Theorem 1.2.2, this completes the proof of this step and hence of the statement.

□

10.3.0.1. Auxiliary lemmata used in the proof of Theorem 10.3.2

Lemma 10.3.3. *For any i, j, the function $\lambda \mapsto U(T^\lambda)_{i,j}$ is continuous on I.*

Proof. By the Induction-Reduction theorem and the definition of U, formula (25) implies that for any $\lambda_1, \lambda_2 \in I$ such that $\lambda_1 + R_k\lambda_2 \in I$ for any $k \in \mathbb{K}$, it holds

$$U(T^{\lambda_1 + R_k\lambda_2}) = \Big(A \circ U(T^{\lambda_1}) \otimes U(T^{\lambda_2}) \circ A^*\Big)_{k,k} \qquad \forall k \in \mathbb{K}. \tag{26}$$

Explicitly computing (26) with $k = e$ yields[3]

$$U(T^{\lambda_1+\lambda_2})_{i,j} = U(T^{\lambda_1})_{i,j} U(T^{\lambda_2})^T_{i,j}. \tag{27}$$

Fix $\lambda_0 \in I$ and choose an open set V such that

- $\int_U U(\lambda_2)^T \, d\lambda_2 > 0$;
- there exists a neighborhood W of λ_0 such that $U + \lambda \subset I$ for any $\lambda \in W$.

This is possible since we can assume $f, g \not\equiv 0$, which yields $U \not\equiv 0$, and the set I is open dense. Then, integrating (27) over V w.r.t. λ_2 yields

$$U(T^\lambda)_{i,j} = \frac{\int_{V+\lambda} U(T^{\lambda_2})_{i,j} \, d\lambda_2}{\int_V U(T^{\lambda_2})^T_{i,j} \, d\lambda_2} \qquad \forall \lambda \in W.$$

Since the function on the r.h.s. is clearly continuous on W this proves the continuity at λ_0 of $U(T^\lambda)$, completing the proof. □

Lemma 10.3.4. *The function $\lambda \mapsto U(T^\lambda)$ can be extended to a continuous function on $\widehat{\mathbb{H}} \setminus \{\hat{o}\}$. Moreover, for any $\lambda_1, \lambda_2 \neq \hat{o}$ it holds $\hat{f}(T^{\lambda_1} \otimes T^{\lambda_2}) \circ U(T^{\lambda_1}) \otimes U(T^{\lambda_2}) = \hat{g}(T^{\lambda_1} \otimes T^{\lambda_2})$.*

[3] For $k \neq 0$ the formula becomes

$$U(T^{\lambda_1+R_k\lambda_2})_{i,j} = U(T^{\lambda_1})_{i,j} U(T^{\lambda_2})^T_{i-k,j-k}.$$

Proof. Let $\lambda_0 \notin I$. Since I is an open and dense set, this implies that λ_0 is in its closure and that we can choose $\lambda_1, \lambda_2 \in I$ such that $\lambda_0 = \lambda_1 + R_{k_0}\lambda_2$ for some $k_0 \in \mathbb{K}$ and $\lambda_1 + R_k\lambda_2 \in I$ for any $k \neq k_0$. We then let

$$U(T^{\lambda_0}) := \Big(A \circ U(T^{\lambda_1}) \otimes U(T^{\lambda_2}) \circ A^*\Big)_{k_0,k_0} \qquad \text{for } \lambda_0 = \lambda_1 + R_{k_0}\lambda_2, \tag{28}$$

We now prove that the above definition does not depend on the choice of λ_1, λ_2 and k_0. By opennes of I, there exists a neighborhood V of λ_2 entirely contained in I. Then, up to taking a smaller V, it holds that $\lambda_1 + R_{k_0}\lambda_2' \in I$ for any $\lambda_2' \in V \setminus \{\lambda_2\}$. By (26), this implies that for any $\mu_1 + R_\ell\mu_2 = \lambda_0$ it holds $(A \circ U(T^{\lambda_1}) \otimes U(T^{\lambda_2'}) \circ A^*)_{k_0,k_0} = (A \circ U(T^{\mu_1}) \otimes U(T^{\mu_2'}) \circ A^*)_{\ell,\ell}$ for λ_2' and μ_2' sufficiently near, but different, to λ_2 and μ_2, respectively. By continuity of U on I, proved in Lemma 10.3.3, this implies that this equation has to hold also for $\lambda_2' = \lambda_2$ and $\mu_2' = \mu_2$. Hence, (28) does not depend on the choice of λ_1, λ_2 and k_0.

Finally, the fact that $\hat{f}(T^{\lambda_1} \otimes T^{\lambda_2}) \circ U(T^{\lambda_1}) \otimes U(T^{\lambda_2}) = \hat{g}(T^{\lambda_1} \otimes T^{\lambda_2})$ for any λ_1, λ_2 follows from (28) and the Induction-Reduction theorem. □

Lemma 10.3.5. *There exists $a \in \mathbb{G}$ such that $U(T^\lambda) = T^\lambda(a)$ for any $\lambda \in \widehat{H} \setminus \{\hat{o}\}$.*

Proof. By definition of U it holds that

$$\bigoplus_{k\in\mathbb{K}} U(T^{\lambda_1+R_k\lambda_2}) \circ A = A \circ U(T^{\lambda_1}) \otimes U(T^{\lambda_2}) \qquad \forall \lambda_1, \lambda_2 \neq \hat{o}.$$

Then, for any i, j, ℓ, k,

$$U(T^{\lambda_1})_{\ell,i} U(T^{\lambda_2})_{\ell-k,j} = \begin{cases} U(T^{\lambda_1+R_k\lambda_2})_{\ell,i} & \text{if } j = i-k, \\ 0 & \text{otherwise.} \end{cases} \tag{29}$$

Since $U(T^{\lambda_1})$ is invertible, there exists i_0 such that $U(T^{\lambda_1})_{e,i_0} \neq 0$. Using (29) one obtains that $U(T^{\lambda_2})_{-k,j} = 0$ for any $j \neq i_0 - k$. Namely, we have proved that $U(T^{\lambda_1})_{-k,\cdot} = \varphi_{-k}(\lambda_1)e_{i_0-k}$ for any h for some $\varphi_{-k} : \mathbb{H} \setminus \{\hat{o}\} \to \mathbb{C}$.

We can rephrase the above result as $U(T^\lambda) = \operatorname{diag}_k \varphi_k(\ell)\, S^{i_0}$. Thus, by the explicit expression of the representation T^λ, in order to complete the proof it suffices to prove that $\varphi_k(\lambda) = \lambda(R_k h_0)$ for some $h_0 \in \mathbb{H}$.

By continuity and unitariety of U, the φ_h's are continuous and satisfy $|\varphi_h(\lambda)| = 1$. Using again (29) with $j = i_0 - k$, we obtain

$$\varphi_\ell(\lambda_1 + R_k\lambda_2) = \varphi_\ell(\lambda_1)\varphi_{\ell-k}(\lambda_2), \qquad \text{for any } \lambda_1, \lambda_2 \neq \hat{o} \text{ and } \ell, k \in \mathbb{K}. \tag{30}$$

In particular, choosing $k = e$ and $\lambda_2 = -\lambda_1$ in the above shows that φ_ℓ can be extended at $\hat{o}$. If $\hat{o}$ is an accumulation point this extension is continuous, as one can see letting $k = e$ and $\lambda_2 \to \hat{o}$ in (30). Then, (30) with $k = e$ implies φ_ℓ is a character of $\widehat{\mathbb{H}}$. By Pontryiagin duality, there exists $h_\ell \in \mathbb{H}$ such that $\varphi(\lambda) = \lambda(h_\ell)$. Finally, by (30) with $k \in \mathbb{K}$ one obtains that $R_{-k}h_\ell = h_{\ell-k}$, which proves that there exists $h_0 \in \mathbb{H}$ such that $\varphi_\ell(\lambda) = R_\ell h_0$. This completes the proof of the statement. □

11. Bispectral invariants for almost-periodic functions

Let $\mathbb{G}$ be a MAP group, in the sense of Section 5. Consider the set $B_2(\mathbb{G})$ of Besicovitch almost periodic functions. Since $\sigma^* : L^2(\mathbb{G}^\flat) \to B_2(\mathbb{G})$ is an isomorphism of Hilbert spaces, we define the spectral invariants of $f \in B_2(\mathbb{G})$ as $I^1_f := I^1_{f'}$ and $I^2_f := I^2_{f'}$, where $f' \in L^2(\mathbb{G}^\flat)$ is such that $f' \circ \sigma = f$.

Since $L^2(\mathbb{G}^\flat)$ is the space of square integrable functions, one could be induced to think the (weak) completness of the invariants on $B_2(\mathbb{G})$ functions to be a consequence of the results of Section 10.2.1. However this is not true, due to the lack of separability of $B_2(\mathbb{G})$ and to the fact that $\mathbb{G}^\flat$ is much bigger[4] than $\mathbb{G}$.

Let $\mathcal{E}$ be the image under σ^* of the set $\mathcal{G}$ defined in (22), that is

$$\mathcal{E} = \left\{ \sigma^* f' \mid f' \in L^2(\mathbb{G}^\flat) \text{ and } \widehat{f'}(T) \text{ is invertible for all } T \in \widehat{\mathbb{G}^\flat} \right\}.$$

We then have the following.

Theorem 11.0.6. *Let $\mathbb{G}$ be a non-compact group. Then, the bispectral invariants are not complete on $B_2(\mathbb{G})$ nor they are complete on $\mathcal{E}$.*

Proof. The main observation is that since $\mathbb{G}$ is non-compact, it holds that $\mathbb{G} \ncong \mathbb{G}^\flat$, and in particular $\mathbb{G}^\flat \setminus \sigma(\mathbb{G}) \neq \varnothing$. Indeed, consider $f', g' \in L^2(\mathbb{G}^\flat)$ be such that $f' = \Lambda^\flat(\xi) g$, where $\Lambda^\flat$ is the left regular representation on $\mathbb{G}^\flat$. Then, $f := f' \circ \sigma$ and $g := g' \circ \sigma$ satisfy $I^2_f = I^2_g$. However, if $\xi \in \mathbb{G}^\flat \setminus \mathbb{G}$ they cannot be deduced via $\Lambda = \Lambda^\flat \circ \sigma$, the left regular representation of $\mathbb{G}$. This proves that bispectral invariants cannot be complete on $B_2(\mathbb{G})$. To complete the proof it suffices to observe that if $f' \circ \sigma \in \mathcal{E}$ then $g' \circ \sigma \in \mathcal{E}$, since $\widehat{g'}(T) = \widehat{f'} \circ R(\xi)$. □

Remark 11.0.7. *The above proof actually shows that the bispectral invariants do not discriminate on any subset of $B_2(\mathbb{G})$ containing $f = f' \circ \sigma$ and $g = g' \circ \sigma$ such that $f' = \Lambda^\flat(\xi) g'$ for some $\xi \in \mathbb{G}^\flat \setminus \sigma(\mathbb{G})$.*

Regarding weak completeness of the invariants, let us restrict to the case $\mathbb{G} = \mathbb{H} \rtimes \mathbb{K}$ considered in Section 10.3 (which taking $\mathbb{K} = \{e\}$ contains the case of an abelian $\mathbb{G}$). As already mentioned, $\mathbb{G}$ is a MAP group and $\mathbb{G}^\flat = \mathbb{H}^\flat \rtimes \mathbb{K}$, where the action of $\mathbb{K}$ is obtained by density of the injection of $\mathbb{H}$ in $\mathbb{H}^\flat$. In this case functions $f \in B_2(\mathbb{G})$ are exactly those such that $f(\cdot, k) \in B_2(\mathbb{H})$ for any $k \in \mathbb{K}$.

Recall also that the unitary irreducible representations of $\mathbb{G}^\flat$ are in bijection with those of $\mathbb{G}$ and are parametrized by $\lambda \in \widehat{\mathbb{H}^\flat} = \widehat{\mathbb{H}}_d$ and $\hat{k} \in \widehat{\mathbb{K}}$. Observe that the topology w.r.t. the λ variable is the discrete one.

We now describe the natural subsets of $B_2(\mathbb{G})$ that we consider for the weak completeness. Fix a bispectrally admissible set $E \subset \widehat{\mathbb{H}}_d$ and decompose it as $E = F \cup G$ as in Definition 5.1.2. Then, consider the set

$$\mathcal{G}_E = \left\{ \sigma^* f' \in B_2(\mathbb{G}) \mid \operatorname{supp} \mathcal{F}(f'(\cdot, k)) \subset E \quad \forall k \in \mathbb{K} \text{ and } \widehat{f'}(T^\lambda) \text{ is invertible } \forall \lambda \in F \right\}. \tag{31}$$

[4] For example, observe that the measure of $\sigma(\mathbb{G})$ in $\mathbb{G}^\flat$ is zero.

Depending on the structure of E, problems can arise even in this case:

Proposition 11.0.8. *Let $\mathcal{G}_E \subset B_2(\mathbb{G})$ be the set defined in (31) and corresponding to the bispectrally admissible set E. Then, if E is a subgroup of $\widehat{\mathbb{H}}_d$ with at least one accumulation point in $\widehat{\mathbb{H}}$ w.r.t. the usual topology, the bispectral invariants are not complete on $\mathcal{G}$.*

In particular, if the cardinality N of $\mathbb{K}$ is even, this happens for all E's satisfying the assumptions of Proposition 6.2.3.

Proof. Since E is a dense subgroup of a locally compact abelian group, by Pontryagin duality it obviously hold $\mathbb{H} \subset \widehat{E}$. However, due to the presence of an accumulation point, the discrete topology of $E \subset \widehat{\mathbb{H}}_d$ is finer than that induced by $\widehat{\mathbb{H}}$. Thus, by [4] there exist $\chi \in \widehat{E} \setminus \mathbb{H}$.

Given any $f = \sigma^* f' \in \mathcal{G}$, define $g = \sigma^* g'$ letting $g'(\cdot, k)$ be the inverse Fourier transform of $\lambda \mapsto \chi(\lambda)\mathcal{F}(f'(\cdot, k)))(\lambda)$, for any $k \in \mathbb{K}$. By definition, $\hat{g}'(T^\lambda) = \chi(\lambda)\hat{f}'(T^\lambda)$. Since $\chi(\lambda) \neq 0$ everywhere, this implies that $\hat{g}'(T^\lambda)$ is invertible for $\lambda \in F$ and hence that $g \in \mathcal{G}$. Moreover, using the fact that χ is a character of E, from Propositions 2.0.7 and 10.3.1 follows that $I^2_f = I^2_g$. Finally, since χ is not a character of $\widehat{H}$, we have that $g \neq \pi_{B_2}(a)f$ for any $a \in \mathbb{G}$, which completes the proof. □

12. Bispectral invariants for lifts

Let us consider $\mathbb{G} = \mathbb{H} \rtimes \mathbb{K}$ as in Section 2. In this section we will discuss bispectral invariants on range L, where L is one of the lift operators described in Section II.

Henceforth, to lighten the notation, when an injective lift is fixed and only functions in range L are considered, we denote the invariants in $L^2(\mathbb{G})$ of Lf, $f \in L^2(\mathbb{H})$, by I^1_f and I^2_f.

12.1. Regular cyclic lifts

Let $L = P \circ \Phi : \mathcal{A} \to L^2(\mathbb{G})$ be a regular cyclic lift where $\mathcal{A} \subset L^2(\mathbb{H})$ is closed w.r.t. the quasi-regular representation π, as introduced in Section 8. Let $\Psi \in L^2(\mathbb{H})$ be the associated wavelet, whose existence is assured by Corollary 8.3.3, and assume that the centering Φ be w.r.t. to the whole $\mathbb{H}$.

Proposition 12.1.1. *Assume that the cardinality N of $\mathbb{K}$ is odd. Let $f \in \mathcal{A}$. Then, for any $\lambda, \lambda_1, \lambda_2 \in \widehat{\mathbb{H}} \setminus \{\hat{o}\}$ and $i, j, k, \ell \in \mathbb{K}$, it holds that*

$$I^1_f(T^\lambda)_{i,j} = \overline{\omega_\Psi(\lambda)_i}\, \omega_\Psi(\lambda)_j \left\langle \omega_{\Phi(f)}(\lambda), S^{i-j}\omega_{\Phi(f)}(\lambda) \right\rangle,$$

$$(A \circ I^2_f(T^{\lambda_1} \otimes T^{\lambda_2}) \circ A^*)_{k,\ell,i,j} =$$
$$\overline{\omega_\Psi(\lambda_1)_i}\,\overline{\omega_\Psi(\lambda_2)_{i-\ell}}\, \omega_\Psi(\lambda_1 + R_k\lambda_2)_j \left\langle \omega_{\Phi(f)}(\lambda_1) \odot S^{\ell-k}\omega_{\Phi(f)}(R_k\lambda_2), S^{i-j}\omega_{\Phi(f)}(\lambda_1 + R_k\lambda_2) \right\rangle.$$

Here, A is the equivalence from $L^2(\mathbb{K} \times \mathbb{K})$ to $\bigoplus_{k\in\mathbb{K}} L^2(\mathbb{K})$ defined in (3) and $\odot$ denote term-wise multiplication.

Proof. Due to the special form of cyclic lifts, it suffices to replace f with $\Phi(f)$ in the expressions of the invariants $\tilde{I}^1_f = I^1_{Pf}$ and $\tilde{I}^2_f = I^2_{Pf}$, corresponding to the almost left-invariant lift associated with L.

By Proposition 8.2.1, direct computations yield

$$\begin{aligned}\tilde{I}^1_f(T^\lambda)_{i,j} &= \overline{\omega_\Psi(\lambda)_i}\,\omega_\Psi(\lambda)_j \sum_{h\in\mathbb{K}} \omega_f(\lambda)_{2h-i}\,\overline{\omega_f(\lambda)_{2h-j}} \\ &= \overline{\omega_\Psi(\lambda)_i}\,\omega_\Psi(\lambda)_j \sum_{\ell\in\mathbb{K}} \omega_f(\lambda)_\ell\,\overline{\omega_f(\lambda)_{\ell+i-j}},\end{aligned}$$

from which the first statement follows. Here, we used the fact that N is even, since it ensured that $\{2h - i \mid h \in \mathbb{K}\} = \mathbb{K}$.

The second statement is proved in a similar way, using also Corollary 8.2.2, Proposition 10.3.1 and making the same change of variables as above. □

As an immediate consequence of the previous result we obtain:

Corollary 12.1.2. *Assume the cardinality N of $\mathbb{K}$ to be odd and that $\hat{\Psi} \neq 0$ a.e. on $\widehat{\mathbb{H}}$. Then, for any $f, g \in \mathcal{A}$, the fact that $I^1_f = I^1_g$ is equivalent to*

$$\left\langle \omega_{\Phi(f)}(\lambda), S^h\omega_{\Phi(f)}(\lambda)\right\rangle = \left\langle \omega_{\Phi(g)}(\lambda), S^h\omega_{\Phi(g)}(\lambda)\right\rangle \text{ for a.e. } \lambda \in \widehat{H}\setminus\{\hat{o}\} \text{ and } h \in \mathbb{K}. \quad (32)$$

Moreover, the fact that $I^2_f = I^2_g$ is equivalent to the fact that, for a.e. $\lambda_1, \lambda_2 \in \widehat{H}\setminus\{\hat{o}\}$ and every $k, h \in \mathbb{H}$ it holds

$$\left\langle \omega_{\Phi(f)}(\lambda_1) \odot S^k\omega_{\Phi(f)}(\lambda_2), S^h\omega_{\Phi(f)}(\lambda_1+\lambda_2)\right\rangle = \left\langle \omega_{\Phi(g)}(\lambda_1) \odot S^k\omega_{\Phi(g)}(\lambda_2), S^h\omega_{\Phi(g)}(\lambda_1+\lambda_2)\right\rangle \quad (33)$$

Finally due to the good properties of the Fourier transforms of cyclically lifted functions, we have the following.

Proposition 12.1.3. *Let $\mathbb{K} = \mathbb{Z}_N$ with N odd. Moreover, assume L to be a regular cyclic lift such that the associated wavelet satisfies $\hat{\Psi} \neq 0$ a.e. on $\widehat{\mathbb{H}}$. Then, the bispectral invariants evaluated on lifted functions are weakly complete on $L^2(\mathbb{H})$.*

Proof. The result essentially follows from Theorem 10.3.2. Indeed, by (17), the parity of N, and the injectivity of L, it suffices to show that there exists a residual subset $\mathcal{G}$ of $L^2(\mathbb{H})$ such that $\widehat{Lf}(T^\lambda)$ is invertible on an open and dense subset of $\widehat{H}\setminus\{\hat{0}\}$. Indeed, this will trivially imply that $\mathcal{G}$ is residual in $\mathcal{A}$. From Proposition 8.2.3 it follows that the set $\mathcal{C}$ of weakly-cyclic $L^2(\mathbb{H})$ functions has this property and is residual in $\mathcal{G}$ by Theorem 6.1.1. □

12.2. Regular left-invariant lifts

Let $L : L^2(\mathbb{H}) \to L^2(\mathbb{G})$ be a regular left-invariant lift and let $\Psi \in L^2(\mathbb{H})$ be the associated wavelet given by Theorem 7.0.16.

Proposition 12.2.1. *Let $f \in L^2(\mathbb{H})$. Then, for any $\lambda, \lambda_1, \lambda_2 \in \widehat{\mathbb{H}} \setminus \{\hat{o}\}$, it holds that*

$$I^1_f(T^\lambda) = \|\omega_f(\lambda)\|^2_{L^2(\mathbb{K})} \overline{\omega_\Psi(\lambda)}^* \otimes \overline{\omega_\Psi(\lambda)},$$

$$(A \circ I^2_f(T^{\lambda_1} \otimes T^{\lambda_2}) \circ A^*)_{k,\ell} = \langle \omega_f(\lambda_1) \odot \omega_f(R_\ell \lambda_2), \omega_f(\lambda_1 + R_\ell \lambda_2)\rangle \; \overline{\omega_\Psi(\lambda_1 + R_k\lambda_2)}^* \otimes \overline{\omega_\Psi(\lambda_1)} \odot \overline{\omega_\Psi(R_k\lambda_2)}.$$

Here, $A = (A_k)_{k\in\mathbb{K}}$ is the equivalence from $L^2(\mathbb{K}\times\mathbb{K})$ to $\bigoplus_{k\in\mathbb{K}} L^2(\mathbb{K})$ defined in (3) and $\odot$ denote term-wise multiplication.

Proof. The first part of the statement follows directly from Proposition 7.2.1 and the properties of the tensor product.

Let us prove the second statement. A simple manipulation, using the Induction-Reduction Theorem, yields

$$\left(A \circ I^2_f(T^{\lambda_1} \otimes T^{\lambda_2}) \circ A\right)_{k,\ell} = \left(A \circ \widehat{Lf}(T^{\lambda_1}) \otimes \widehat{Lf}(T^{\lambda_2}) \circ A^*\right)_{k,\ell} \circ \widehat{Lf}(T^{\lambda_1 + R_\ell\lambda_2})^*. \quad (34)$$

Since $A_\ell(\overline{\omega_f(\lambda_1)} \otimes \overline{\omega_f(\lambda_2)}) = \overline{\omega_f(\lambda_1)} \odot \overline{\omega_f(R_\ell\lambda_2)}$, by Proposition 7.2.1 the first part of the above is

$$\begin{aligned}\left(A \circ \widehat{Lf}(T^{\lambda_1}) \otimes \widehat{Lf}(T^{\lambda_2}) \circ A^*\right)_{k,\ell} &= \left(\left(A(\overline{\omega_f(\lambda_1)} \otimes \overline{\omega_f(\lambda_2)})\right)^* \otimes \left(A(\overline{\omega_\Psi(\lambda_1)} \otimes \overline{\omega_\Psi(\lambda_2)})\right)\right)_{k,\ell} \\ &= \left(A_k(\overline{\omega_f(\lambda_1)} \otimes \overline{\omega_f(\lambda_2)})\right)^* \otimes \left(A_\ell(\overline{\omega_\Psi(\lambda_1)} \otimes \overline{\omega_\Psi(\lambda_2)})\right) \\ &= (\overline{\omega_f(\lambda_1)} \odot \overline{\omega_f(R_\ell\lambda_2)})^* \otimes (\overline{\omega_\Psi(\lambda_1)} \odot \overline{\omega_\Psi(R_k\lambda_2)}).\end{aligned} \quad (35)$$

On the other hand, Proposition 7.2.1 immediately yields

$$\widehat{Lf}(T^{\lambda_1 + R_\ell\lambda_2})^* = \overline{\omega_\Psi(\lambda_1 + R_\ell\lambda_2)}^* \otimes \overline{\omega_f(\lambda_1 + R_\ell\lambda_2)}. \quad (36)$$

Since $(\overline{\omega_f(\lambda_1)} \odot \overline{\omega_f(R_\ell\lambda_2)})^* \overline{\omega_f(\lambda_1 + R_\ell\lambda_2)} = \langle \omega_f(\lambda_1) \odot \omega_f(R_\ell\lambda_2), \omega_f(\lambda_1 + R_\ell\lambda_2)\rangle$, putting together (34), (35) and (36) proves the statement. ☐

Recall that $A_k \omega_\Psi(\lambda_1) \otimes \omega_\Psi(\lambda_2) = \omega_\Psi(\lambda_1) \odot \omega_\Psi(R_k\lambda_2)$. We then have the following.

Corollary 12.2.2. *Let $\Psi \in L^2(\mathbb{H})$ be a weakly admissible wavelet. Then, for any $f, g \in L^2(\mathbb{H})$, the fact that $I^1_f = I^1_g$ is equivalent to*

$$\|\omega_f(\lambda)\|_{L^2(\mathbb{K})} = \|\omega_f(\lambda)\|_{L^2(\mathbb{K})} \text{ for a.e. } \lambda \in \widehat{H} \setminus \{\hat{o}\}.$$

If moreover Ψ is such that $\omega_\Psi(\lambda_1) \odot \omega_\Psi(\lambda_2) \neq 0$ for a.e. λ_1, λ_2, then the fact that $I^2_f = I^2_g$ is equivalent to

$$\langle \omega_f(\lambda_1) \odot \omega_f(\lambda_2), \omega_f(\lambda_1 + \lambda_2)\rangle = \langle \omega_g(\lambda_1) \odot \omega_g(\lambda_2), \omega_g(\lambda_1 + \lambda_2)\rangle \text{ for a.e. } \lambda_1, \lambda_2 \in \widehat{H} \setminus \{\hat{o}\}. \quad (37)$$

Proof. By Proposition 12.2.1, the first statement is equivalent to $\overline{\omega_\Psi(\lambda)}^* \otimes \overline{\omega_\Psi(\lambda)} \neq 0$ for a.e. $\lambda \in \widehat{H} \setminus \{\hat{o}\}$. Since this is equivalent to $\omega_\Psi(\lambda) \neq 0$, by Theorem 4.0.4 this is true for any weakly admissible vector.

The proof of the second statement is similar, after making the change of variables $R_k\lambda_2 \mapsto \lambda_2$. Indeed, the requirements are then $\omega_\Psi(\lambda_1 + \lambda_2) \neq 0$, which is satisfied by weak admissibility, and $\omega_\Psi(\lambda_1) \odot \omega_\Psi(\lambda_2) \neq 0$, which is satisfied by assumption. ☐

12.2.0.2. Trace invariants In this section we show how, exploiting Proposition 12.2.1, one can actually decrease the set of invariants.

Let us recall that the trace of a trace class operator C acting on the Hilbert space $\mathcal{H}$ is defined as

$$\operatorname{Tr} C = \sum_i \langle Ce_i, e_i \rangle,$$

where $\{e_i\}_i$ is a basis of $\mathcal{H}$. Being the product of two Hilbert-Schimdt operators, the bispectral invariants are of trace class.

Definition 12.2.3. *The* trace bispectral invariants *associated with the regular left-invariant lift L of $f \in L^2(\mathbb{H})$ are the elements of the set*

$$\operatorname{Tr} I_f^2 = \{\operatorname{Tr} I_f^2(T^{\lambda_1}, T^{\lambda_2}) \mid \lambda_1, \lambda_2 \in \widehat{H} \setminus \{\hat{o}\}\}.$$

Corollary 12.2.4. *Let $f \in L^2(\mathbb{H})$. Then, for any $\lambda_1, \lambda_2 \in \widehat{H} \setminus \{\hat{o}\}$ it holds that*

$$\operatorname{Tr} I_f^2(\lambda_1, \lambda_2) =$$
$$\sum_{k \in \mathbb{K}} \langle \omega_f(\lambda_1) \odot \omega_f(R_k\lambda_2), \omega_f(\lambda_1 + R_k\lambda_2) \rangle \operatorname{Tr} \left(\overline{\omega_\Psi(\lambda_1 + R_k\lambda_2)}^* \otimes \overline{\omega_\Psi(\lambda_1)} \odot \overline{\omega_\Psi(R_k\lambda_2)} \right)$$

Proof. The statement is an immediate consequence of Proposition 12.2.1 and of the similarity-invariance of the trace. □

For any $\Psi \in L^2(\mathbb{H})$, any $\lambda_1, \lambda_2 \in \widehat{H} \setminus \{\hat{o}\}$, and any $h \in \mathbb{K}$ we define the function $B_\Psi^h \in L^2(\mathbb{K})$ as

$$B_\Psi^h(k) = \omega_\Psi(\lambda_1 + R_k\lambda_2)(h) \, \overline{\omega_\Psi(\lambda_1)}(h) \, \overline{\omega_\Psi(R_k\lambda_2)}(h).$$

In particular,

$$\operatorname{Tr} \left(\overline{\omega_\Psi(\lambda_1 + R_k\lambda_2)}^* \otimes \overline{\omega_\Psi(\lambda_1)} \odot \overline{\omega_\Psi(R_k\lambda_2)} \right) = \sum_{h \in \mathbb{K}} B_\Psi^h(k). \tag{38}$$

This justifies the following.

Definition 12.2.5. *A wavelet $\Psi \in L^2(\mathbb{H})$ is* trace admissible *if $\omega_\Psi(\lambda_1) \odot \omega_\Psi(R_k\lambda_2) \neq 0$ and the family $\{B_\Psi^h\}_{h \in \mathbb{K}}$ is a basis of $L^2(\mathbb{K})$ for a.e. $\lambda_1, \lambda_2 \in \widehat{H} \setminus \{\hat{o}\}$.*

Observe that a trace admissible wavelet is always weakly admissible and satisfies the assumptions of Corollary 12.2.2.

Proposition 12.2.6. *Let L be a regular left-invariant lift with associated wavelet $\Psi \in L^2(\mathbb{H})$. Then, if Ψ is trace admissible it holds $\operatorname{Tr} I_f^2 \supset I_f^2$.*

Proof. Since Ψ satisfies the assumptions of Corollary 12.2.2, we only need to show that $\operatorname{Tr} I_f^2 = \operatorname{Tr} I_g^2$ if and only if (37) is satisfied.

Putting together Corollary 12.2.4 and (38), exchanging the summation order we have

$$\begin{aligned}\operatorname{Tr} I^2_f(\lambda_1,\lambda_2) &= \sum_{h\in\mathbb{K}}\sum_{k\in\mathbb{K}} B^h_\Psi(k)\langle\omega_f(\lambda_1)\odot\omega_f(R_k\lambda_2),\omega_f(\lambda_1+R_k\lambda_2)\rangle \\ &= \sum_{h\in\mathbb{H}}\left\langle k\mapsto\langle\omega_f(\lambda_1)\odot\omega_f(R_k\lambda_2),\omega_f(\lambda_1+R_k\lambda_2)\rangle,\overline{B^h_\Psi}\right\rangle.\end{aligned}$$

Since $\{B^h_\Psi\}_{h\in\mathbb{K}}$ is a basis of $L^2(\mathbb{K})$, this shows that $\operatorname{Tr} I^2_f(\lambda_1,\lambda_2) = \operatorname{Tr} I^2_g(\lambda_1,\lambda_2)$ if and only if for any $k\in\mathbb{K}$

$$\langle\omega_f(\lambda_1)\odot\omega_f(R_k\lambda_2),\omega_f(\lambda_1+R_k\lambda_2)\rangle = \langle\omega_g(\lambda_1)\odot\omega_g(R_k\lambda_2),\omega_g(\lambda_1+R_k\lambda_2)\rangle.$$

This completes the proof. □

13. Rotational bispectral invariants for left-invariant lifts modulo the action of $\mathbb{H}$

Let $\mathbb{G}=\mathbb{H}\rtimes\mathbb{K}$ be as in Section 2, with $\mathbb{K}$ finite with N elements. We pose the following.

Definition 13.0.7. *The* rotational power spectrum invariants *of $f\in L^2(\mathbb{G})$ are the set $J^1_f=\{J^1_f(\lambda,k)\mid\lambda\in\widehat{\mathbb{H}}\setminus\{\hat o\}$ and $k\in\mathbb{K}\}$ such that*

$$J^1_f(\lambda,k) \coloneqq \hat f(T^{R_k\lambda})\circ\hat f(T^\lambda)^*.$$

The rotational bispectral invariants *of $f\in L^2(\mathbb{G})$ are the set $J^2_f=\{J^2_f(\lambda_1,\lambda_2,k)\mid\lambda_1,\lambda_2\in\widehat{\mathbb{H}}\setminus\{\hat o\}$ and $k\in\mathbb{K}\}$ such that*

$$J^2_f(\lambda_1,\lambda_2,k)\coloneqq\hat f(T^{R_k\lambda_1})\otimes\hat f(T^{\lambda_2})\circ\hat f(T^{\lambda_1}\otimes T^{\lambda_2})^*.$$

For any $k\neq e$, the above defined quantities are invariant only w.r.t. the action of $\mathbb{K}$ on $\mathbb{G}$. This implies that they can only discriminate up to the action of $\mathbb{K}$.

Since $J^2_f\supset I^2_f$, as a consequence of Theorem 10.3.2 we immediately obtain the following.

Corollary 13.0.8. *Rotational bispectral invariants are complete w.r.t. the action of $\mathbb{K}$ on the set $\mathcal{G}$ defined in (20). Namely, for any $f,g\in\mathcal{G}$ it holds that*

$$J^2_f=J^2_g \iff f=R_kg\quad\text{for some } k\in\mathbb{K}.$$

Let $L=P\circ\Phi: L^2(\mathbb{H})\to L^2(\mathbb{G})$ be the composition of a regular left-invariant lift P and a centering $\Phi:\mathcal{A}\to\mathcal{A}$ w.r.t. $U\subset\mathbb{H}$, see Definition 2.0.4. Denote by $\Psi\in L^2(\mathbb{H})$ the wavelet associated with P, given by Theorem 7.0.16.

In the following, for $f\in L^2(\mathbb{H})$ we let $J^1_f=J^1_{Lf}$ and $J^2_f=J^2_{Lf}$. The following can be proved as Proposition 12.2.1.

Proposition 13.0.9. *Let $f \in L^2(\mathbb{H})$. Then, for any $\lambda, \lambda_1, \lambda_2 \in \widehat{H} \setminus \{\hat{o}\}$ and any $k, h, \ell \in \mathbb{K}$ it holds*

$$J^1_f(\lambda, k) = \langle S^{-k}\omega_{\Phi(f)}(\lambda), \omega_{\Phi(f)}(\lambda)\rangle_{L^2(\mathbb{K})} \overline{\omega_\Psi(\lambda)}^* \otimes \overline{\omega_\Psi(R_k\lambda)},$$

$$(A \circ J^2_f(\lambda_1, \lambda_2, k) \circ A^*)_{h,\ell} =$$

$$\langle \omega_{\Phi(f)}(\lambda_1) \odot \omega_{\Phi(f)}(R_{k+\ell}\lambda_2), \omega_{\Phi(f)}(\lambda_1 + R_\ell\lambda_2)\rangle \overline{\omega_\Psi(\lambda_1 + R_h\lambda_2)}^* \otimes \overline{\omega_\Psi(\lambda_1)} \odot \overline{\omega_\Psi(R_{h+k}\lambda_2)}.$$

Here, A is the equivalence from $L^2(\mathbb{K} \times \mathbb{K})$ to $\bigoplus_{h\in\mathbb{K}} L^2(\mathbb{K})$ defined in (3) and $\odot$ denote term-wise multiplication.

Corollary 13.0.10. *Let $\Psi \in L^2(\mathbb{H})$ be a weakly admissible wavelet. Then, for any $f, g \in L^2(\mathbb{H})$, the fact that $J^1_f = J^1_g$ is equivalent to*

$$\left\langle \omega_{\Phi(f)}(\lambda), S^h\omega_{\Phi(f)}(\lambda)\right\rangle = \left\langle \omega_{\Phi(g)}(\lambda), S^h\omega_{\Phi(g)}(\lambda)\right\rangle \text{ for a.e. } \lambda \in \widehat{H} \setminus \{\hat{o}\} \text{ and } h \in \mathbb{K}. \quad (39)$$

If moreover Ψ is such that $\omega_\Psi(\lambda_1) \odot \omega_\Psi(R_k\lambda_2) \neq 0$ for any $k \in \mathbb{K}$ and a.e. λ_1, λ_2, then the fact that $J^2_f = J^2_g$ is equivalent to

$$\left\langle \omega_{\Phi(f)}(\lambda_1) \odot S^h\omega_{\Phi(f)}(\lambda_2), \omega_{\Phi(f)}(\lambda_1 + \lambda_2)\right\rangle = \left\langle \omega_{\Phi(g)}(\lambda_1) \odot S^h\omega_{\Phi(g)}(\lambda_2), \omega_{\Phi(g)}(\lambda_1 + \lambda_2)\right\rangle \quad (40)$$

Remark 13.0.11. *Although in the above result no assumptions on the cardinality of $\mathbb{K}$ are required, comparing it with to Corollary 12.1.2 seems to suggest that rotational bispectral invariants carry less information than the bispectral invariants of cyclic lifts. Indeed, while (39) is identical to (32), in (40) we consider N invariants for couple (λ_1, λ_2) against the N^2 of (33).*

The rest of this section is devoted to prove the weak completeness of the rotational bispectral invariants in this context.

Theorem 13.0.12. *Let the lift $L = P \circ \Phi : \mathcal{A} \to L^2(\mathbb{G})$ be the composition of a regular left invariant lift $P : L^2(\mathbb{H}) \to L^2(\mathbb{G})$ and a centering $\Phi : \mathcal{A} \to \mathcal{A}$ w.r.t. $U \subset \mathbb{H}$. Moreover, assume the wavelet Ψ associated with P to be weakly cyclic and such that $\hat{\Psi} \neq 0$ a.e..*

Then, the rotational bispectral invariants evaluated on lifted functions are complete on the set $\{f \in \mathcal{C} \cap \mathcal{A} \mid \hat{f} \neq 0$ a.e. on $\widehat{\mathbb{H}}\}$ w.r.t. the action of elements in $U \times \mathbb{K}$. Here, $\mathcal{C}$ is the set of weakly-cyclic functions on $L^2(\mathbb{H})$ introduced in Definition 3.0.3.

Namely, for any couple $f, g \in \mathcal{A}$ of weakly cyclic functions such that $\hat{f}$ and $\hat{g} \neq 0$ a.e. on $\widehat{\mathbb{H}}$, it holds

$$J^2_f = J^2_g \iff f = \pi(a)g \quad \text{for some } a \in U \times \mathbb{K} \subset \mathbb{G}.$$

Proof. Since Φ is a centering w.r.t. U, by the properties of the abelian Fourier transform w.r.t. translations follows that if f is weakly-cyclic so is $\Phi(f)$. Then the statement is equivalent to the fact that for any couple f, g of weakly-cyclic functions, $J^2_{Pf} = J^2_{Pg}$ if and only if $f = R_k g$ for some $k \in \mathbb{K}$.

Let us consider two weakly-cyclic functions such that $J^2_{Pf} = J^2_{Pg}$ and denote by $I \subset \widehat{\mathbb{H}}$ the set

$$I = \{\lambda \mid \det \operatorname{Circ} \omega_f(\lambda) \neq 0 \text{ and } \det \operatorname{Circ} \omega_g(\lambda) \neq 0\}.$$

By the weak-cyclicity of f and g this set is open and dense.

The proof follows similar steps as the proof of Theorem 10.3.2. One has however to pay additional care, due to the non-invertibility of the lifted Fourier transforms. The most delicate point is the commutation with the tensor product, which was proved in one line in step 1 of Theorem 10.3.2. Here we delay the proof of this fact to Lemma 13.0.14.

- *Step 1.1 - Definition of the candidate quasi-representation U on T^λ for $\lambda \in I$:* For any $\lambda \in I$ we let

  $$U(T^\lambda)^* = \operatorname{Circ} \omega_g(\lambda) \left(\operatorname{Circ} \omega_f(\lambda)\right)^{-1} \qquad \forall \lambda \in I.$$

  By the definition it is obvious that $U(T^\lambda)$ is circulant. In particular, the definition implies that $U(T^\lambda)^* S^k \omega_f(\lambda) = S^k \omega_g(\lambda)$ for any $k \in \mathbb{K}$. Moreover, $U(T^\lambda)$ is unitary by the equality of the rotational power invariant and (39) of Corollary 13.0.10.

  By the expression of the Fourier transform of Pf given in Proposition 7.2.1, the definition of U is also equivalent to

  $$\widehat{Lf}(T^{R_k\lambda}) \circ U(T^\lambda) = \widehat{Lg}(T^{R_k\lambda}) \qquad \forall \lambda \in I, \forall k \in \mathbb{K}.$$

  As a consequence, $\lambda \mapsto U(T^\lambda)$ is constant on the orbits $\{R_k\lambda\}_{k\in\mathbb{K}}$ of λ.

- *Step 1.2 - Definition of U on $T^{\lambda_1} \otimes T^{\lambda_2}$:* To extend the definition of U to the tensor product of representations we use the Induction-Reduction Theorem. Let us call $I^\otimes$ the set of couples $(\lambda_1, \lambda_2) \in \widehat{\mathbb{H}} \times \widehat{\mathbb{H}}$ such that $\lambda_1 + R_k\lambda_2 \in I$ for any $k \in \mathbb{K}$. Then, we let

  $$U\left(T^{\lambda_1} \otimes T^{\lambda_2}\right) = A^* \circ \left(\bigoplus_{k\in\mathbb{K}} U(T^{\lambda_1 + R_k\lambda_2})\right) \circ A \qquad \forall (\lambda_1, \lambda_2) \in I^\otimes. \tag{41}$$

  By the corresponding property of $\lambda \mapsto U(T^\lambda)$, this definition implies that $(\lambda_1, \lambda_2) \mapsto U(T^{\lambda_1} \otimes T^{\lambda_2})$ is constant on the orbits $\{(R_k\lambda_1, R_k\lambda_2)\}_{k\in\mathbb{K}}$ of (λ_1, λ_2).

  By the Induction-Reduction Theorem and the properties of the Fourier transform, (41) is equivalent to

  $$\widehat{Lf}(T^{R_k\lambda_1} \otimes T^{R_k\lambda_2}) \circ U(T^{\lambda_1} \otimes T^{\lambda_2}) = \widehat{Lg}(T^{R_k\lambda_1} \otimes T^{R_k\lambda_2}) \qquad \forall (\lambda_1, \lambda_2) \in I^\otimes, \forall k \in \mathbb{K}. \tag{42}$$

- *Step 1.3 - It holds that $U(T^{\lambda_1} \otimes T_2^\lambda) = U(T^{\lambda_1}) \otimes U(T_2^\lambda)$:* This is proved in Lemma 13.0.14.

- *Step 2 - The function $\lambda \mapsto U(T^\lambda)$ is continuous on I:* Since $\lambda \mapsto \omega_f(\lambda)$ and $\lambda \mapsto \omega_g(\lambda)$ are measurable on I, so it is $\lambda \mapsto U(T^\lambda)$. The same arguments used in Lemma 10.3.3 can be then used to prove the continuity.

- *Step 3 - The function $\lambda \mapsto U(T^\lambda)$ can be extended to a continuous function on $\widehat{\mathbb{H}} \setminus \{\hat{o}\}$. Moreover, the function $(\lambda_1, \lambda_2) \mapsto U(T^{\lambda_1} \otimes T^{\lambda_2})$ defined via (41) on $\widehat{\mathbb{H}} \times \widehat{\mathbb{H}}$ satisfyies (42):* This is proved exactly as in Lemma 10.3.4.

- *Step 4 - There exists $k \in \mathbb{K}$ such that $U(T^\lambda) = T^\lambda(o, k)$ for any $\lambda \in \widehat{\mathbb{H}} \setminus \{\hat{o}\}$:* This is proved with the same arguments as in Lemma 10.3.5. Indeed, the fact that now $\lambda \mapsto U(T^\lambda)$ is constant on the orbits $\{R_k\lambda\}_{k\in\mathbb{K}}$ implies that the φ_k's obtained there have to be independent of k. Since $\varphi_k(\lambda) = R_k x_0$ for some $x_0 \in \mathbb{H}$, this implies that $x_0 = 0$ and hence $\varphi_k \equiv 0$. Obviously this proves that $U(T^\lambda) = S^k = T^\lambda(o, k)$, for some $k \in \mathbb{K}$.

- *Step 5 - It holds that $R_k f = g$:* This follows exactly as in Theorem 10.3.2. □

The above result can be easily adapted to the subspaces of Besicovich almost periodic functions introduced in Section 5.1.

Theorem 13.0.13. *Let $E \subset \widehat{\mathbb{H}}$ be a bispectrally admissible set, $K \subset \mathbb{H}$ be compact and consider a lift $L = P \circ \Phi : \mathcal{A} \to L^2(\mathbb{G})$. Here, $P : B_2(\mathbb{H}) \to B_2(\mathbb{G})$ is a left invariant lift with associated wavelet $\Psi \in \mathcal{T}(E)$ and $\Phi : \mathcal{A} \to \mathcal{A}$ is a centering w.r.t. K. Moreover, assume the wavelet Ψ to be AP-weakly cyclic in $\mathcal{T}(E)$ and such that $a_\Psi(\lambda) \neq 0$ for any $\lambda \in E$.*

Then, the rotational bispectral invariants evaluated on lifted functions are complete on the set $\{f \in \mathcal{C}^{AP} \cap \mathcal{T}(E) \mid a_f(\lambda) \neq 0$ for all $\lambda \in E\}$ w.r.t. the action of elements of $K \times \mathbb{K} \subset \mathbb{G}$. Here, $\mathcal{C}^{AP}$ is the set of AP-weakly cyclic functions on $\mathcal{T}(E)$ introduced in Definition 5.1.1.

Proof. To prove the result it suffices to replay step by step the proof of the previous theorem. The only point where one has to pay attention is step 4. Indeed, the arguments employed there allow to show that $U(T^\lambda) = T^\lambda(o, k)$ for some $k \in \mathbb{K}$ when $\lambda \in E$ is such that either $\lambda = \lambda_1 + R_h\lambda_2$ for some $h \in \mathbb{K}$ and a couple $(\lambda_1, \lambda_2) \in I^\otimes$ or there exists λ' such that $(\lambda, \lambda') \in I^\otimes$. The fact that one of these properties is always satisfied for any $\lambda \in E$ is a consequence of the bispectral invariance of E. □

13.0.1. Auxiliary lemma for the proof of Theorem 13.0.12

Lemma 13.0.14. *Let $(\lambda_1, \lambda_2) \in I^\otimes$ be such that*

1. *$\omega_\Psi(\lambda_1)$, $\omega_\Psi(\lambda_2) \neq 0$,*
2. *for any $k \in \mathbb{K}$ it holds that $\omega_\Psi(\lambda_1) \odot \omega_\Psi(R_k\lambda_2) \neq 0$.*

Then, if $\hat{f}(R_\ell\lambda_2)$ and $\hat{g}(R_\ell\lambda_2) \neq 0$ for all $\ell \in \mathbb{K}$ it holds

$$U(T^{\lambda_1} \otimes T^{\lambda_2}) = U(T^{\lambda_1}) \otimes U(T^{\lambda_2}) \qquad .$$

Proof. Since $U(T^\lambda)$ is circulant, we write $U(T^\lambda) = \sum_{j\in\mathbb{K}} u_j(\lambda) S^j$ where $u(\lambda) \in L^2(\mathbb{K})$. Let $C := A \circ U(T_1^\lambda) \otimes U(T_2^\lambda) \circ A^*$. Then, by the Induction-Reduction Theorem, to complete the proof it suffices to prove that $C = \bigoplus_{k\in\mathbb{K}} U(T^{\lambda_1+R_k\lambda_2})$.

Since a simple computation using Proposition 2.1.2 shows that $(A\circ(S^i\otimes S^j)\circ A^*)_{k,\ell} = \delta_{\ell-k,i-j}S^i$, the block $C_{k,\ell}$ of C is

$$C_{k,\ell} = \sum_j u_j(\lambda_1)u_{j-(\ell-k)}(\lambda_2)S^i.$$

This proves that $C_{k,\ell}$ is circulant and that C is block-circulant, i.e., $C_{k,\ell} = C_{k+\alpha,\ell+\alpha}$.

We claim that, for any $k \in \mathbb{K}$, the vectors $\{\omega_f(\lambda_1) \odot \omega_f(R_{k+\alpha}\lambda_2)\}_{\alpha\in\mathbb{K}}$ form a basis of $L^2(\mathbb{K})$. Indeed, letting V be the matrix with α-th row $\omega_f(\lambda_1) \odot \omega_f(R_{k+\alpha}\lambda_2)$, we have

$$V = \operatorname{diag}(\omega_f(\lambda_1))\ \operatorname{Circ}(\omega_f(R_k\lambda_2)).$$

The fact that $\omega_f(\lambda)$ is cyclic and $\hat{f}(R_{-\ell}\lambda) \neq 0$ for any $\ell \in \mathbb{K}$, implies that V is invertible and thus proves the claim.

The previous claim allows to define the operator $\bar{C}_k \in \mathcal{L}(L^2(\mathbb{K}))$ as

$$\bar{C}_k\left(\omega_f(\lambda_1)\odot\omega_f(R_{k+\alpha}\lambda_2)\right) = \sum_\ell C_{k,\ell}\left(\omega_f(\lambda_1)\odot\omega_f(R_{k+\alpha+\ell}\lambda_2)\right) \qquad \forall\alpha\in\mathbb{K}.$$

By the explicit expression of the Fourier transform given in Proposition 7.2.1 and Corollary 7.2.3, and the assumptions on the wavelet Ψ, the equality of the second invariants implies

$$\left\langle \left(U(T^{\lambda_1+R_k\lambda_2})^*\circ C_{k,\cdot} - \pi_k\right) A\left(\omega_f(\lambda_1)\otimes\omega_f(R_\alpha\lambda_2)\right), \omega_f(\lambda_1+R_k\lambda_2)\right\rangle = 0 \qquad \forall k,\alpha\in\mathbb{K}.$$

Here we denote by $C_{k,\cdot} \in \mathcal{L}(\bigoplus_{h\in\mathbb{K}} L^2(\mathbb{K}), L^2(\mathbb{K}))$ the k-th "row-block" operator of C. Clearly, this is equivalent to

$$\left\langle \left(U(T^{\lambda_1+R_k\lambda_2})^*\circ \bar{C}_k - \mathrm{Id}\right) A_k\left(\omega_f(\lambda_1)\otimes\omega_f(R_\alpha\lambda_2)\right), \omega_f(\lambda_1+R_k\lambda_2)\right\rangle = 0 \qquad \forall k,\alpha\in\mathbb{K}.$$

Moreover, recall that $U(T^\lambda) = U(T^{R_h\lambda})$ for any $h \in \mathbb{K}$. Thus making the change of variables $(\lambda_1,\lambda_2) \mapsto (R_n\lambda_1, R_n\lambda_2)$ in the above we obtain that for any $k, n, \alpha \in \mathbb{K}$ it holds

$$\left\langle \left(U(T^{\lambda_1+R_k\lambda_2})^*\circ \bar{C}_k - \mathrm{Id}\right) A_k\left(\omega_f(R_n\lambda_1)\otimes\omega_f(R_{n+\alpha}\lambda_2)\right), S^{-n}\omega_f(\lambda_1+R_k\lambda_2)\right\rangle = 0.$$

Observe that for any $k, n \in \mathbb{K}$, it holds that $\operatorname{span}_{\alpha\in\mathbb{K}}\{A_k\left(\omega_f(R_n\lambda_1)\otimes\omega_f(R_{n+\alpha}\lambda_2)\right)\} = L^2(\mathbb{K})$. Thus, the above is equivalent to

$$\operatorname{range}\left(U(T^{\lambda_1+R_k\lambda_2})^*\circ\bar{C}_k - \mathrm{Id}\right) \perp S^{-n}\omega_f(\lambda_1+R_k\lambda_2).$$

Since $\omega_f(\lambda_1+R_k\lambda_2)$, is cyclic it then follows that $U(T^{\lambda_1+R_k\lambda_2})^*\circ\bar{C}_k = \mathrm{Id}$ for any $k\in\mathbb{K}$. By Lemma 13.0.15 this implies that

$$C_{k,\ell} = \begin{cases} U(T^{\lambda_1+R_k\lambda_2}) & \text{if } k=\ell,\\ 0 & \text{otherwise,}\end{cases}$$

completing the proof of the statement.

□

Lemma 13.0.15. *Let $v, w \in L^2(\mathbb{K})$ be cyclic vectors and $e = A(v \otimes w)$. Moreover, let $C \in \mathcal{L}(L^2(\mathbb{K}))$ be the operator defined on the basis $e_j = A_j(v \otimes w)$ as*

$$Ce_j = \sum_{k \in \mathbb{K}} C_{-k} e_{k-j},$$

where C_{-k} are circulant operators on $L^2(\mathbb{K})$. Then $C = \mathrm{Id}$ if and only if $C_0 = \mathrm{Id}$ and $C_{-k} = 0$ for any $k \neq 0$.

Proof. The sufficient part of the statement is obvious. Let us assume that $C = \mathrm{Id}$ and define

$$\mathfrak{C} = \sum_{k \in \mathbb{K}} \left(\bigoplus_{h \in \mathbb{K}} C_{-k} \right) \tilde{S}^k \in \mathcal{L}\left(\bigoplus_{h \in \mathbb{K}} L^2(\mathbb{K}) \right).$$

A simple computation shows that $\mathfrak{C}_{j,\ell} = C_{\ell - j}$, which implies that $\mathfrak{C}e = (Ce_j)_j = e$.

Since C_{-k} is circulant, for any $k \in \mathbb{K}$ there exists $c^{-k} \in L^2(\mathbb{K})$ such that $C_{-k} = \sum_{j \in \mathbb{K}} c_j^{-k} S^j$. Thus, by Proposition 2.1.2 and the fact that $e = A(v \otimes w)$, we obtain

$$\begin{aligned} v \otimes w &= A^* \circ \mathfrak{C} \circ A(v \otimes w) \\ &= \sum_{k,j \in \mathbb{K}} c_j^{-k} A^* \circ \left(\bigoplus_{h \in \mathbb{K}} S^j \right) \circ \tilde{S}^k \circ A(v \otimes w) \\ &= \sum_{k,j \in \mathbb{K}} c_j^{-k} S^{j+k} \otimes S^j (v \otimes w). \end{aligned}$$

By cyclicity of v and w, $\{(S^\alpha \otimes S^\beta)(v \otimes w)\}_{\alpha,\beta \in \mathbb{K}}$ is a basis of $L^2(\mathbb{K}) \otimes L^2(\mathbb{K})$, and thus the above implies

$$\sum_{k,j \in \mathbb{K}} c_j^{-k} S^{j+k} \otimes S^j = \mathrm{Id}\,.$$

Finally, this is equivalent to $c_e^e = 1$ and $c_j^{-k} = 0$ if j or $k \neq e$, which proves the statement. □

13.1. Real valued functions

As discussed in Section 3.1, when the action of $\mathbb{K}$ is *even* (see Definition 3.1.1) it holds that $\mathcal{C} \cap L^2_{\mathbb{R}}(\mathbb{R}^2) = \varnothing$. That is, Theorem 13.0.12 gives no information on real valued functions. In this section we will show how to exploit the tools introduced in Section 3.1 to obtain the completeness for real valued functions.

Theorem 13.1.1. *Let the assumptions of Theorem 13.0.12 to be satisfied. Moreover, assume $\mathcal{A} \subset L^2_{\mathbb{R}}(\mathbb{H})$, and that the wavelet Ψ is real valued.*

Then, the rotational bispectral invariants evaluated on lifted functions are weakly complete on $\mathcal{A}$ w.r.t. the action of elements of $U \times \mathbb{K} \subset \mathbb{G}$. Here, $\mathcal{C}_{\mathbb{R}}$ is the set of weakly $\mathbb{R}$-cyclic functions introduced in Definition 3.1.3.

Proof. If $\mathbb{K}$ is not even, since $\mathcal{C}_{\mathbb{R}} = \mathcal{C} \cap L^2_{\mathbb{R}}(\mathbb{H})$, the result is simply a restatement of Theorem 13.0.12. Thus we only need to prove the result for $\mathbb{K}$ even.

Recall the notations introduced in Section 3.1 and define

$$\mathcal{Y} = \left\{ v \in L^2(\mathbb{K}) \mid v(h) = -\overline{v(h+k_0)} \quad \forall h \in \mathbb{K} \right\}.$$

Considering the realification of $L^2(\mathbb{K})$, it splits $\mathbb{R}$-orthogonally as $L^2(\mathbb{K}) \cong \mathcal{X} \oplus \mathcal{Y}$.

We need the following observations:

- From the invariance w.r.t. the shifts of $\mathcal{X}$ it follows that the equivalence A restricts to an equivalence between $\mathcal{X} \otimes \mathcal{X}$ and $\bigoplus_{k\in\mathbb{K}} \mathcal{X}$. This allows us to define $A_{\mathbb{R}} = B^{-1} \circ A \circ B$.
- From Proposition 7.2.1, for any $\lambda \in \widehat{\mathbb{H}}$, it follows that $\ker \widehat{Lf}(T^\lambda) \supset \mathcal{Y}$ and that $\mathcal{X}$ is an invariant subspace for $\widehat{Lf}(T^\lambda)$. Thus we define $\widehat{Lf}_{\mathbb{R}}(T^\lambda) = B^{-1} \circ \widehat{Lf}(T^\lambda) \circ B$.

Let $f, g \in \mathcal{C}_{\mathbb{R}} \cap \mathcal{A}$ satisfying the conditions in the statement. To complete the proof it now suffices to show that there exists $k \in \mathbb{K}$ such that $\widehat{Lf}_{\mathbb{R}}(T^\lambda) \circ S^k_{\mathbb{R}} = \widehat{Lg}_{\mathbb{R}}(T^\lambda)$ for all $\lambda \in \widehat{\mathbb{H}} \setminus \{\hat{o}\}$. Indeed, since $\ker \widehat{Lf}(T^\lambda) \supset \mathcal{Y}$ and $S(\mathcal{Y}) = \mathcal{Y}$, this implies that $\widehat{Lf}(T^\lambda) \circ S^k = \widehat{Lg}(T^\lambda)$.

To this aim, let $I \subset \widehat{\mathbb{H}}$ be the set of λ's such that $\operatorname{Circ}_{\mathbb{R}} B^{-1}\omega_f(\lambda)$ and $\operatorname{Circ}_{\mathbb{R}} B^{-1}\omega_g(\lambda)$ are invertible. We then let

$$U_{\mathbb{R}}(T^\lambda) = \operatorname{Circ}_{\mathbb{R}} B^{-1}\omega_g(\lambda) \left(\operatorname{Circ}_{\mathbb{R}} B^{-1}\omega_f(\lambda) \right)^{-1} \qquad \forall \lambda \in I.$$

Let also $I^{\otimes}$ to be the set of couples $(\lambda_1, \lambda_2) \in \widehat{\mathbb{H}} \times \widehat{\mathbb{H}}$ such that $\lambda_1 + R_k\lambda_2 \in I$ for any $k \in \mathbb{K}$ and define

$$U_{\mathbb{R}}(T^{\lambda_1} \otimes T^{\lambda_2}) = A^*_{\mathbb{R}} \circ \left(\bigoplus_{k\in\mathbb{K}} U_{\mathbb{R}}(T^{\lambda_1 + R_k\lambda_2}) \right) \circ A_{\mathbb{R}} \qquad \forall (\lambda_1, \lambda_2) \in I^{\otimes}.$$

With these definitions to obtain that $\widehat{Lf}_{\mathbb{R}}(T^\lambda) \circ S^k_{\mathbb{R}} = \widehat{Lg}_{\mathbb{R}}(T^\lambda)$ when the rotational bispectral invariants of f and g coincide it suffices to replay the exact same arguments of Theorem 13.0.12, substituting $S_{\mathbb{R}}$, $A_{\mathbb{R}}$, $\operatorname{Circ}_{\mathbb{R}}$ and $\widehat{Lf}_{\mathbb{R}}$ to S, A, Circ and $\widehat{Lf}$, respectively. □

As for Theorem 13.0.12, the above theorem can be easily adapted to $\mathcal{T}(E) \subset B_2(\mathbb{H})$. Observe that, for $f \in \mathcal{T}(E)$ be real-valued it is necessary that $E = -E$.

Theorem 13.1.2. *Let the assumptions of Theorem 13.0.13 to be satisfied. Moreover, assume $E = -E$, and that the wavelet Ψ is real valued.*

Then, the rotational bispectral invariants evaluated on lifted functions are complete on the set $\{f \in \mathcal{C}^{AP}_{\mathbb{R}} \cap \mathcal{T}(E) \mid f$ is real-valued and $a_f(\lambda) \neq 0$ for all $\lambda \in E\}$ w.r.t. the action of elements of $K \times \mathbb{K} \subset \mathbb{G}$. Here, $\mathcal{C}^{AP}_{\mathbb{R}}$ is the set of AP-weakly $\mathbb{R}$-cyclic functions introduced in Definition 5.1.1.

13.2. Compactly supported real-valued functions on $\mathbb{R}^2$

In this section we particularize and extend the results of the previous section to the case of $\mathbb{G} = SE(2, N)$ and to real-valued compactly supported functions on the plane, introduced in Section 6.1. The natural choice for a lift is $L = P \circ \Phi_c : \mathcal{V}(D_r) \to L^2(SE(2, N))$, where P is a regular left-invariant lift with a real valued associated wavelet, while $\Phi_c : \mathcal{V}(D_r) \to \mathcal{V}(D_R)$ is the centering operator defined in Section 6.1

Recall that in this context $\mathbb{Z}_N$ is even if N is even and not even if N is odd. The main theorem of this section is the following.

Theorem 13.2.1. *Assume the wavelet Ψ to be real valued, weakly $\mathbb{R}$-cyclic, and such that $\hat{\Psi} \neq 0$ a.e. on $\widehat{\mathbb{R}^2}$. Then, two weakly $\mathbb{R}$-cyclic functions in $\mathcal{V}(D_R)$ can be deduced via the action of $SE(2, N)$ if these two conditions are satisfied*

- *their bispectral invariants coincide a.e. (i.e., $I^2_f(\lambda_1, \lambda_2) = I^2_g(\lambda_1, \lambda_2)$ for a.e. (λ_1, λ_2));*
- *their rotational bispectral invariants coincide on an open set (i.e., $J^2_f(\lambda_1, \lambda_2, k) = J^2_g(\lambda_1, \lambda_2, k)$ for any $k \in \mathbb{Z}_N$ and (λ_1, λ_2) in an open set).*

In particular, the rotational bispectral invariants evaluated on lifted functions are weakly complete on $\mathcal{V}(D_R) \cap L^2_{\mathbb{R}}(\mathbb{R}^2)$ and discriminate on an open and dense set.

Proof. The fact that $\mathcal{C}_{\mathbb{R}}$ is an open dense subset of $\mathcal{V}(D_R)$ is proved in Theorem 6.1.1. The result then follows from Theorems 13.0.12 and 13.1.1. Indeed, the Fourier transform $\hat{f}$ of $f \in \mathcal{V}(D_R)$ is analytic and hence $\hat{f} \neq 0$ on an open and dense set. We then proceed exactly as in the proofs of Theorems 13.0.12 and 13.1.1, observing that $\lambda \mapsto U(T^\lambda)$ is now an analytic function. Since the rotational bispectral invariants coincide only on an open set $V \subset \widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}^2}$, the commutation with the tensor product of step 1.3 holds only on there. However, this allows to jump directly to step 4 and prove that there exists $k \in \mathbb{Z}_N$ such that $U(T^\lambda) \equiv S^k$ for any λ in a section of V. Then, $U(T^\lambda) \equiv S^k$ everywhere by analyticity, and the proof can be concluded. □

13.3. Almost-periodic functions on the plane

Let us fix a countable bispectrally invariant set E such that $E = -E$ and consider the set $\mathcal{T}(E) \subset B_2(\mathbb{R}^2)$ of almost-periodic functions with frequencies in E, as introduced in Section 6.2.1. Moreover, let us consider a $P : B_2(\mathbb{R}^2) \to B_2(SE(2, N))$ left-invariant lift with associated wavelet $\Psi \in \mathcal{T}(E)$, a compact $K \subset \mathbb{R}^2$ and the centering operator $\Phi : \mathcal{R}_K \to \mathcal{R}_K$ defined in Section 6.2.2. We recall that the left-invariant lift P is obtained from a left-invariant lift $P' : L^2((\mathbb{R}^2)^\flat) \to L^2(SE(2, N)^\flat)$ via the isomorphism $\sigma^* : L^2((\mathbb{R}^2)^\flat) \to B_2(\mathbb{R}^2)$ defined in Section 5.

Theorem 13.3.1. *Assume the wavelet $\Psi \in \mathcal{T}_{\mathbb{R}}(E)$ is AP-weakly $\mathbb{R}$-cyclic and such that $a_\Psi(\lambda) \neq 0$ for all $\lambda \in E$. The rotational bispectral invariants evaluated on lifted functions are weakly complete on $\mathcal{T}_{\mathbb{R}}(E)$ w.r.t. the action of elements of $K \times \mathbb{Z}_N \subset SE(2, N)$.*

Proof. By Theorem 13.1.2 it suffices to show that the set

$$\{f \in \mathcal{C}^{\mathrm{AP}}_{\mathbb{R}} \cap \mathcal{T}_{\mathbb{R}}(E) \mid a_f(\lambda) \neq 0 \text{ for all } \lambda \in E\}$$

is residual in $\mathcal{T}_{\mathbb{R}}(E)$. This follows from Theorem 6.2.1 and the fact that $\{f \in \mathcal{T}_{\mathbb{R}}(E) \mid a_f(\lambda) \neq 0 \text{ for all } \lambda \in E\}$ is residual. Indeed, this follows immediately by countability of E and the fact that $\mathbb{C} \setminus \{0\}$ is open and dense. □

Remark 13.3.2. *From the proof of the previous theorem it follows that, when E is finite, the rotational bispectral invariants discriminate on an open and dense set.*

References

[1] K DeLeeuw and I. Glicksberg, *Almost periodic functions on semigroups*, Acta Math. **105** (1961), no. 1-2, 99–140.

[2] J. Dixmier, *C^*-Algebras*, North-Holland publishing company, 1977.

[3] H Führ and M. Mayer, *Continuous wavelet transforms from semidirect products : Cyclic representations and Plancherel measure*, J. Fourier Anal. Appl. (2002), 1–23.

[4] E. Hewitt, *A remark on characters of locally compact Abelian groups*, Fundam. Math. **53** (1964), 55–64.

[5] E. Hewitt and KA Ross, *Abstract harmonic analysis - Volume 1*, Springer-Verlag, 1963.

[6] Herbert Heyer, *Groups with Chu duality*, Probab. Inf. Theory II (1973).

[7] L. Nachbin, *The Haar integral*, 1965.

[8] Fethi Smach, Cedric Lemaître, Jean-paul Gauthier, Johel Miteran, and Mohamed Atri, *Generalized Fourier Descriptors with Applications to Objects Recognition in SVM Context*, J. Math. Imaging Vis. **30** (2008), 43–71.